Actual Infinity and God

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Abstract

Greek thought gave birth to the two notions of infinity that dominated the history of philosophy and mathematics: the potential infinite *and the* actual infinite*. Initially infinity was understood in the literal sense of one, another one, yet another one – and so on* indefinitely*, without an end, endlessly, infinitely. The discovery of the whole-parts relation by the school of Parmenides mediated the turn "inwards" – introducing* infinite divisibility*. This development is related to the contribution made by Zeno's paradoxes: in particular, the bisecting paradox and the flying arrow. Aristotle developed a theory of continuity based upon two criteria that formally match the Weierstrass-Cantor-Dedekind understanding continuity –in spite of the fact that Aristotle rejected the actual infinite while Weierstrass-Cantor-Dedekind accepted it as the basis for their view of continuity. Mathematics faced it first crisis when the Pythagorean arithmetization was confronted with the discovery of irrational numbers. Aristotle's objections to the actual infinite turned out to be hall-marks of infinity. The prevalent Greek view was that since the* apeiron *(the unbounded-unlimited) was formless, God (as thought thinking itself) cannot be infinite (formless). Gregor von Nyssa (fourth century AD) was the first thinker who positively asserted that God is infinite. Augustine provided a starting point for Cantor because he contrasted succession with simultaneity: God can oversee an infinity of numbers at once, whereas the human mind can only contemplate a multiplicity in succession. (Maimon formulated a similar view in 1792.) Nicholas of Cusa distinguishes between the absolute (actual) infinity of God and the endlessness of reality – in God all oppositions coincide (God is the*

coincidentia oppositorum*). Compare the view of al-Ghazālī mentioned by Verhoef* et.al. *It turned out that Aristotle's objections to the actual infinite are in fact characteristic features of infinity. (It is shown with reference to the smallest transfinite ordinal number, ω). According to Descartes the infinite is perfect and the finite is imperfect. Yet the potential infinite governed mathematics until the 19th century. In his letter of July 12, 1831 to Schumacher Gauss stated that "in this manner I protest against the use of an infinite magnitude as something completed, which is never allowed in mathematics". The limit concept gave rise to the second foundational crisis of mathematics – irrational numbers cannot be defined as the limits of converging sequences of rational numbers. Weierstrass, Cantor and Dedekind once more introduced the idea of the actual infinite. Cantor distinguishes between the* potential infinite*, the* actual infinite *and the* absolute infinite *(God). Hermann Weyl restricted infinity to the potential infinite, while leaving actual infinity open for God, for him God is the completed infinite. In line with the historical contours outlined, the idea of* eternity *also entered the theological domain in the form of two apparently opposing notions: eternity as an* endless period *of time, and eternity as* timelessness*. The line runs from Parmenides, Plotinus (Enneads III/7), Boethius, Kierkegaard (the* nunc aeternum*/the eternal now) and Schilder. Is infinity brought into mathematics on a Christian theological foundation? The important distinction between conceptual knowledge and concept-transcending knowledge (idea-knowledge) is introduced. At the same time Bernays rejected the attempt to reduce continuity to discreteness. Erasmus and Verhoef captures the actual infinite in familiar terms: an infinite totality, a completed whole with all members present all at once. The potential infinite and the actual infinite should rather be designated as the* successive infinite *and the at* once infinite*. In addition, a closer analysis of the notion of a "totality" and its connection to "all at once" and "(actual) infinity" was undertaken.*

In most cultures in which a particular number system is developed, it is fairly easy to count at least to ten. Spengler takes it a step further by claiming that every culture has its own *number world* entailing that number as such does not exist. We only have diverse *number types*, such as an Indian, Arabic, Antique and an "abendländischen Zahlentypus" (number type of the West) (Spengler, 1923-I:78-9). Yet no number system and number type can avoid the implicit or explicit use of the most basic ordinal numbers employed in counting a few items.

Although it may seem straightforward to define mathematics with reference to two aspects of reality, namely *number* and *space*, modern mathematicians rather speak about a discipline investigating *formal systems*. The logicism of Russell even claims that instead of *quantity* mathematics has to account for ***order***. In 1833 W. Hamilton already described algebra as the “science of pure time or order in progression” (see Cassirer, 1957:85). This view continues the conviction, originally advocated by Leibniz, that the notion of “an ordinal number is logically prior to cardinal number”, and, more generally, “that mathematics may be defined, in Leibnizian fashion, as the science of order” (see Smart, 1958:245). Maddy explains the difference between ordinal and cardinal numbers as follows: “Cardinal numbers tell ‘how many’ – one, two, three ... – as opposed to ordinal numbers, which tell ‘how many-ith’ – first, second, third, ...” (Maddy, 1997:17).

1. Progressions and endlessness

The underlying problem concerns *the* relationship between the *one and the many* – understood in two different senses. The first instance observes a multiplicity accessible in succession and the second multiplicity is present as a whole not given in succession but *at once*.

The famous mathematician and logician, Kurt Gödel, employs the terms “plurality” and “unity”. Wang remarks that Gödel in his distinction between sets and concepts (and classes), is talking about *sets*, but when he speaks of classes as “pluralities of things” (Gödel, 1988:220) the phrase is consistent with his later use of the word ‘class’ to the extent that such pluralities form sets only if they are *unities*.

2. Succession and discreteness

On the one hand we therefore have successions and on the other pluralities (multiplicities) given at once, as a whole. In a different context Russell criticizes Bolzano for not distinguishing the “many from the whole which they form” (Russell, 1956:70).

This means that number is ultimately concerned with discrete (successive) progressions. Clearly, Gödel’s terminology continues to reflect the above-mentioned problem of the one-and-the-many.

The first kind, namely a succession, is the closest to our natural ability to count: 1, 2, 3, 4, . . . This “count-ability” was eventually captured in the concept (*d*)*enumerability*. Yet, establishing the "how many" of a multiplicity implicitly makes an appeal to the *numerical order of succession* which is a *primitive* notion in mathematics. That is to say it is *indefinable*. Weyl affirms this point: "Taking the most primitive object of mathematics, the sequence of natural numbers, …" (Weyl, 1932:78).

One or another order or succession is needed to operate with a multiplicity in the absence of articulated ordinal numbers. Suppose a traditional cultural group with a number system limited to ten receive a travel association with 29 members. How are they going to serve meals to the group. The answer needs the idea of a one-to-one mapping. Ask the tour guests to stand in a straight line and make an "x" in front of each one – then the meals prepared could successively be matched one-by-one.

3. Infinity and the nature of number

The order of succession manifest in ordinary acts of counting points at the equally *indefinability* of "greater and less" – as acknowledged by Russell (see Russell, 1956:194; see also page 167). He also remarks that "progressions are the very essence of discreteness" (Russell, 1956:299). And discreteness captures the core meaning of number.

It is therefore strange that works on the foundations of set theory and dealing with the philosophy of mathematics often refer to mathematics as "the science of formal systems". To those who are inclined to an axiomatic approach this statement means the same as "mathematics is set theory" (cf. Meschkowski, 1972b:356).

In spite of its recent origin as a mathematical discipline, set theory has from the very start been confronted with basic trends running through the history of mathematics. We only have to refer to the tension between the so-called *uncompleted infinitude* and *completed infinitude* – a contrast which has been familiar since Greek philosophy in terms of the opposition between the *potential infinite* and the *actual infinite*. The uncompleted infinite is used to indicate the conception that the infinite is literally in-finite, i.e. one, another one, without an end, endlessly, *infinitely*. The completed infinite, again, is seen as a quantity which is determined in all its parts while it simultaneously exceeds every finite quantity.

Early Greek philosophy already wrestled with the nature of succession. The number zero was supposed not to be a number, but was rather seen as the *origin* of number. In the case of the Pythagoreans number reached divine status, captured in the belief that *everything is number*. Similar to this view we also find other attempts to account for an original element, such as water, air or fire.

The striking choice of Anaximander is what he designated as the principle of origin, namely the *apeiron*, that is, the unlimited/unbounded/infinite. In short, *endlessness* or *infinity*. It certainly does include our above-mentioned most basic understanding of endlessness – one, another one, and so on, indefinitely, infinitely. The concept of a mere succession is used in an extensive sense, literally beyond all limits. But it appeared that Zeno, an early philosopher from the school of Parmenides, who is known for his arguments against *multiplicity* and *motion* – such as the bisecting paradox and the race between Achilles and the tortoise. See Aristotle's *Physics* (cf. 233a13ff. and 239b5ff.).

4. The whole-parts relation

We still have access to a few significant original Fragments of the pre-Socratic philosophers, collected by two German scholars, Hermann Diels and Walther Kranz. The B Fragments are the authentic ones. B Fr.3 of Zeno is arguably the first discovery of the *whole parts relation*.

The peculiar significance of his third Fragment is found in the fact that it explicitly explores both sides of the whole-part relation. Consider his two lines of argumentation, from the parts to the whole and from the whole to the parts:

> When multiplicity exists, then necessarily only as many (things) exist as what are actually there, no more and no less. When there however are as many as what exist, then it (the number thereof) must be limited.

In this first half Zeno therefore argues from multiplicity to limitation. The opposite line is followed in the last half:

> When multiplicity exists, then that which exists (the number thereof) is unlimited. Because continually other ones exist in between those which exist and again others between these. Thus that which is (the number thereof) is unlimited.

Both main parts of this Fragment begin with the phrase, "when multiplicity exists". Yet it arrives at contrasting conclusions – in the first instance the number of existing things is *limited*, and in the second one it is *unlimited*. The static spatial terms which Parmenides and his school use suggest the possibility that Zeno is indeed exploring the two sides of the spatial whole-part relation.

If multiplicity in the initial comment indicates a multiplicity of parts (of the world) then their sum total must be limited while at the same time constituting the world-whole. If, alternatively, one starts with the world-whole to account for the parts, then it would indeed be possible to localize the multiplicity of parts in such a way that there would always be further parts present in between – an argument which of course could be continued indefinitely with regard to all parts.

The discovery of the whole-part relation was therefore indissolubly linked to the development of a notion of infinity in Greek philosophy, since it is concerned with the *infinite divisibility* of the (world-) whole. We shall return to this issue below when we consider how Kant in his *Kritik der reinen Vernunft* wrestled with infinity and the whole-parts relation.

5. Infinity and continuity

Infinity is also linked to continuity. Aristotle says "'what is infinitely divisible is continuous" (*Phys*. 200b19). He proceeds; "it is plain that everything continuous is divisible into divisibles that are infinitely divisible: for if it were divisible into indivisibles, (15) we should have an indivisible in contact with an indivisible, since the extremities of things that are continuous with one another are one and are in contact (*Phys*. 231b14-17).

Anaxagoras understood the nature of spatial continuity in a way that is still relevant today. He says:

> In that which is small there is no smallest, since there always exists something smaller. That which is can never cease to exist through further division, no matter how far we continue this division (B Fr.3). And since no smallest can exist, it also cannot insulate or contain itself, but must, as in the beginning, exist with everything else (B Fr.6).

This *simultaneous existence* suggests the coherence of spatial continuity which includes all (material) things – a continuity which is not, however, the co-ordination of discrete (separated) parts, as if separated with an axe (B Fr.9).

Since Aristotle provided the classical formulation of the notion that a whole is more than its parts (in his *Politeia*:1253a:19-20), this idea has exerted an indelible influence on the history of philosophy and the various special sciences. Before Aristotle someone like Anaximander also highlighted, according to him, an essential characteristic of the infinite, namely that it is that the "*apeiron* is ageless" (B Fr.2) and that "the apeiron is without death and transience" (B Fr.3). Connecting succession and endlessness with an ageless non-transient *apeiron* appears to be problematic, for how can the *apeiron* both be *variable* and *constant*.

Such a position rather reflects the philosophy of Heraclitus with the contrast between becoming and being. He holds that everything changes but nonetheless searches for what is *lasting* amidst all *change*. Frey even speaks about the *tension* between becoming and being in the thought of Heraclitus (Frey, 1968:13).

6. Infinity turned inwards

Zeno's bisection paradox prompted Aristotle to distinguish two kinds of infinity. Moving from point A to point B entails that it is first necessary to traverse half the distance, then half of the remaining distance, and so on indefinitely. Zeno concludes that an infinite number of spatial sub-intervals must be crossed to move from A to B and this is impossible in a finite period of time. It is after all impossible to actually exhaust the infinite. Therefore motion is impossible. As mentioned earlier the most basic meaning of infinity is found in extending any succession of numbers indefinitely, endlessly. Exploring the infinite divisibility of something continuous turns infinite succession inwards but does not exceed the general meaning of successive infinity.

Aristotle confronts Zeno's problem with the following argument:

> In the act of dividing the continuous distance into two halves one point is treated as two, since we make it a starting-point and a finishing-point: and this same result is also produced by the act of reckoning halves as well as by the act of dividing into halves. (25) But if divisions are made in this way, neither the distance nor the motion will be continuous: for motion if it is to be continuous must relate to what is continuous: and though what is continuous contains an infinite number of halves, they are not actual but potential halves (*Physica*, 263 a 23 ff.).

7. Aristotle's objections to the actual infinite

Aristotle does not accept the actual infinite. He raises two objections – see *Physica*, 204 a 20ff., *Metaphysica*, 1066 b 11ff., and *Metaphysica*, 1084 a 1ff.):

i. if the actually infinite consists of parts then these parts must themselves be actually infinite, which would imply the absurdity that the whole is no longer larger than a part [Euclid's second axiom reads: The Whole is larger than a Part]; and

ii. If it consists of finite parts, this would imply the impossibility that the infinite can be counted, or there would have to be transfinite (cardinal) numbers which are neither even nor uneven.

It is understandable, therefore, that the *formative* deity of Aristotle (the *nous* as *thought of thought* – *Metaphysica*, 1074 b 34-35) is *finite*. According to Aristotle, only that which is *limited* can be *known* (*conceived*), and he consequently does not hesitate to conclude from the unlimited nature of matter that matter as such cannot be *known* (*Metaphysica*, 1036 a 8-9).

8. God is not infinite: Plato and Aristotle

One implication of this negative interpretation of matter is that both Plato and Aristotle negate infinity as a predicate of God. Infinity was exclusively ascribed to formless matter (Mühlenberg, 1966:28).

What remains in the thought of Plato and Aristotle is an *Ur-Gegensatz* (original opposition) between *form* and *matter*: Happ remarks: "Hier bleibt – wie bei Platon und in der Akademie – ein Ur-Gegensatz bestehen" (Happ, 1971:805, note 628). Happ continues by mentioning that "Owens exceeds this level when he speaks about a *Pros-hen* unity of the four Aristotelian causes, therefore de facto aiming at reducing everything to 'Form'. One can relate the *Form-Being* ('pros-hen') [focal meaning] to the highest form and the modes of *Matter-Being* to the highest matter, but not the highest matter to the highest form: Here remains – as with Plato and within the Academy – an original dualism in effect" (see Owens, 1951)

Mühlenberg holds that the philosophy of Plato and Aristotle develops by negating *infinity* as a predicate of God because infinity was entirely assigned to matter (Mühlenberg, 1966:28). For Plato the *one* as absolute unity is pure *peras* (delimitation). Nonetheless, the *apeiron* (unlimited) does not originate in the *monas*, because the thought of Plato advances from the split between the limited and the unlimited/infinite. Happ alludes to the two domains in Plato's thought as follows:

> Adjacent to his earlier distinguished two spheres of being, the noetic domain of eternal immutable (ideas) and the sensory perceivable province of *γιγνόμενα* [becoming, being generated], Plato now introduces another hardly comprehensible third domain (Happ, 1971:98).

Mühlenberg holds that Aristotle exclusively places infinity in matter ("das Unendliche ausschlieβlich der Materie zuweist" – Mühlenberg, 1966:28). Mühlenberg establishes that Gregory of Nyssa (335-394) was the first Christian thinker advancing the idea that God is infinite. But this elaboration had to come to terms with the prevailing understanding that God is limited, as it is found in the theology of Origen (184-253 – see Origen's *De Princ*. II,9,1 – Mühlenberg, 1966:26). Aristotle's self-contemplating God would not be able to know itself if it was not limited.

9. The actual infinity of God

Interestingly, more than 150 years ago Georg Cantor, the founder of transfinite arithmetic and set theory, advanced his own view of the actual infinity of God with reference to Origen. Plotinus (204-270), Gregory of Nyssa (335-395) and St. Augustine (354-430) explored key elements of the actual infinite while a few points of departure also penetrated the field of mathematics itself. Let us consider the experiment of playing with infinity in the footsteps of Galileo who contemplated the astounding correlation of the natural numbers and squares – in his dialogue from March 1638.

Not all numbers are squares, such as 1, 4, 9, 16, 25 ... Combining all numbers, i.e. square and non-square numbers, are certainly more than the square numbers on their own. From 0-100 there are only ten squares (100 = 10^2). That means only one tenth are squares; from 0-10000 there are only 100 squares, i.e. 100/10000 = one hundredth; from 0-1000000 there are only a 1000 squares, i.e. one thousandth, and so forth.

10. The whole equivalent to a part

However, if we ask how many square numbers exist, we can answer: as many as there are square roots, since every square has a root and every root has a square. Then, however, there are as many squares as the totality of all numbers!

1^2 2^2 3^2 4^2..........

1 2 3 4 ....

In his posthumously published work on *Paradoxes of the Infinite* (1851) Bernard Bolzano generalized this result by arguing that in the case of infinite sets the *whole* set could be mapped in a one-to-one way to a genuine *subset* of the initial set. This entails that the *whole* is equivalent to a *part*, thus instantiating Aristotle's first objection to the existence of actual infinity, namely that the whole is always *greater* than a *part* (see Bolzano, 1920, par.20:27ff.).

St. Augustine noted that whereas our human understanding can only comprehend a succession of numbers God can oversee all the integers at once as an actually infinite totality (*Ganzes*). Cantor explains it as follows:

> Now that the h. Augustine asserts the total, intuitive perception of the set (*v*) "quodam ineffabili modo", a parte Dei ["in a certain indescribable way", on the part of God], he at the same time *formaliter* recognizes this set as an actual-infinite whole, as a *transfinitum*, and we are forced to follow him in it (Cantor, 1962:402).
>
> [Indem nun der h. Augustin die totale, intuitive Perzeption der Menge (*v*) "quodam ineffabili modo", *a parte Dei* behauptet, erkennt er zugleich diese Menge *formaliter* als ein aktual-unendliches Ganzes, als ein *Transfinitum* an, und wir sind gezwungen, ihm darin zu folgen (Cantor, 1962:402).]

It was Cusanus who completed the circle by relegating the cosmos to *endlessness* to make room for the actual infinity of God as the *coincidentia oppositorum* (the coincidence of opposites). This view dates even further back, namely to the obscure formulation of a later disciple of Heraclitus:

> For all things are alike in that they differ, all harmonize with one another in that they conflict with one another, all converse in that they do not converse, all are rational in being irrational; individual things are by nature contrary, because they mutually agree. For rational world-order [nomos] and nature [physis], by means of which we accomplish all things, do not agree in that they agree (see Diels-Kranz I, 157; Heraclitus, B. Fragm. 30-105).

It should also be remembered that Scriptures does not speak about God's infinity, let alone about the difference between the potential and the actual infinite. Theologians extrapolate God's infinity from his *eternity* and *omnipresence*. The philosophical tradition provides us with two options regarding eternity: *an endless succession of time* or *timelessness*. An endless sequence in time reflects the numerical time order of succession while omnipresence reflects the spatial time order of at once (simultaneity).

The potential infinite and the actual infinite should rather be designated as the *successive infinite* (SI) and the *at once infinite* (AI). In addition, a closer analysis of the notion of a "totality" and its connection to "all at once" and "(actual) infinity" was needed.

11. The successive infinite (*infinitum successivum*) and the at once infinite (*infinitum simultaneum*)

The medieval legacy regarding God's infinity wrestled with the difference between the *successive infinite* (*infinitum successivum*) and the *at once infinite* (*infinitum simultaneum*) (see Maier, 1964:77-79 and Maier, 1949). This means that these two expressions, as just noted, make an appeal to our numerical intuition of *succession* and our spatial intuition of *simultaneity* (*at once*) – in contradistinction to the expressions *potential infinity* and *actual infinity*, that were coined by Aristotle. The crucial difference is that any successive infinity of numbers could be portrayed *as if* they are given at once as an infinite whole or an infinite totality.

Descartes turns the mature conception of ancient Greek philosophy upside down by viewing the infinite as *perfect* and the finite as *imperfect*. We can have an insight (*intelligi*) into the infinite, but we cannot grasp (*comprehendi*) it in a concept.

In his *Kritik der reinen Vernunft* Immanuel Kant continues to defend the successive infinite while rejecting the idea of an *infinite totality*. Note his qualification, namely that that which is successively infinite and never be an *infinite whole* (Ganz = wholeness, i.e., it can never be an infinite totality):

> Notwithstanding this, it is by no means permissible to say of such a whole, which is infinitely divisible, that it consists of an infinite number of parts. For although all the parts are contained in the intuition of the whole, it does not contain the whole division, which consists only in the progressive decomposition, or in the regressus itself, which makes the series real in the first place. Now, since this regress is infinite, all the members (parts) to which it arrives are contained in the given whole as aggregates, but not the whole series of division, which is successively infinite and never whole, consequently not an infinite set, and no compilation of the same can be represented in a whole (Kant, A-1781:524; B-1787:552).
>
> [Diesem ungeachtet ist es doch keineswegs erlaubt, von einem solchen Ganzen, das ins Unendliche teilbar ist, zu sagen: es bestehe aus unendlich viel Teilen. Denn obgleich alle Teile in der Anschauung des Ganzen enthalten sind, so ist doch darin nicht die ganze Teilung enthalten, welche nur in der fortgehenden Dekomposition, oder dem Regressus selbst besteht, der die Reihe allererst wirklich macht. Da dieser Regressus nun unendlich ist, so sind zwar alle Glieder (Teile), zu denen er gelangt, in dem gegebenen Ganzen als Aggregate enthalten, aber nicht die ganze Reihe der Teilung, welche sukzessivunendlich und niemals ganz ist, folglich keine unendliche Menge, und keine Zusammennehmung derselben in einem Ganzen darstellen kann] (This is the only place in the CPR where Kant employs the word "sukzessivunendlich.")

The just quoted section is quite ambiguous regarding the successive infinite divisibility of a whole and the intuition of a whole: “For although all the parts are contained in the intuition of the whole, it does not contain the whole division”. Kant also speaks about a *regressus* without escaping from the ambiguity entailed in the distinction between the infinite regress of all the parts and the *whole* of the series. Although the idea of an infinite totality surfaces in Kant’s thought he ultimately opts for the restriction of infinity to the successive infinite (SI):

> but not the whole series of division, which is successively infinite [sukzessivunendlich] and never whole [a totality], consequently not an infinite set, and no compilation of the same can be represented in a whole.

The underlying distinction between *succession* and *at once* continued to accompany the way in which God’s infinity is portrayed. According to Maimon human perception is bound to time, entailing that only an absolute mind can think any succession of numbers at once, without any passage of time. Therefore, “that which the mind in its limited form considers as a mere idea, is in terms of its absolute existence a real object” (Maimon, 1790:228).

Cantor characterizes the potential and the actual infinite as follows:

> The potential infinite is preferably affirmed where an undetermined, variable magnitude appears, which either increases beyond all finite limits, …, or decreases beyond every finite limit in smallness. … In general I refer to the potential infinite wherever an undetermined magnitude comes into view, which is capable of uncountable many determinations. [Das P.-U. wird vorzugsweise dort ausgesagt, wo eine unbestimmte, *veränderliche endliche* Größe vorkommt, die entweder über alle endlichen Grenzen hinaus wächst (unter diesem Bilde denken wir uns z. B. die sogenannte Zeit, von einem bestimmten Anfangsmomente an gezählt) oder unter jede endliche Grenze der Kleinheit abnimmt (was z. B. die legitime Vorstellung eines sogenannten Differentials ist); allgemeiner spreche ich von einem P.-U. überall da, wo eine *unbestimmte* Größe in Betracht kommt, die unzählig vieler Bestimmungen fähig ist.]
>
> Under an actually infinite, in contrast, is a *Quantum* to be comprehended that on the one hand is *not variable*, but much rather in all its parts firm and determined, a genuine constant, while at once on the other exceeding every finite magnitude of the same kind in size.
> [II. Unter einem A.-U.; … ist dagegen ein Quantum zu verstehen, das einerseits *nicht veränderlich,* sondern vielmehr in allen seinen Teilen fest und bestimmt, eine richtige *Konstante* ist, zugleich aber andrerseits *jede endliche Größe* derselben Art an Größe übertrift (Cantor, 1962:401 and see Strauss, 2019:146).]

12. Defining a set

This legacy was further explored in by Bolzano and in Cantor's *Mengenlehre* (set theory). Cantor defines a *set* as follows:

> By a 'set' we mean any combination M of certain well-distinguished objects in our intuition or our thinking (which are called the 'elements' of M) into a whole.
>
> [Unter einer 'Menge' verstehen wir jede Zusammenfassung *M* von bestimmten wohlunterschiedenen Objekten *m* unsrer Anschauung oder unseres Denkens (welche die 'Elemente' von *M* genannt werden) zu einem Ganzen (Cantor, 1895:481; 1962:282)].

The latter distinguishes between transfinite *ordinal* numbers and *transfinite* cardinal numbers. The smallest transfinite ordinal number, which is designated as *omega* (ω), is simply the set of natural numbers in their natural order of succession. Just recollect our earlier explanation of the difference between ordinal and cardinal numbers (see also Maddy, 1997:17).

The number ω must conform to the following conditions. It must have a first element; each element must have a successor; each element must have a predecessor except for the first one; and there must be no last element.

Without providing the technical detail we can now state that ω could be presented in four different ways: as *even* and *uneven* and at once also *neither* as even *nor* as uneven (see Cantor, 1962:178-179).

Even	$2.\omega$	
Uneven	$1 + 2.\omega$	
Neither even		$\neq \omega.2$
Nor uneven		$\neq \omega.2+1$

These remarkable characteristics should be compared with the notion of Cusanus, namely that God as the *actually infinite* is the union of all opposites: the *coincidentia oppositorum* – he surely did after all recognize something essential about the infinite! For a historical account of the genesis of the expressions *infinitum successivum* and *infinitum simultaneum* see Maier (1964:77-79).

We have already established that up to Gauss mathematics was dominated by the potential infinite. From the perspective of a non-reductionist ontology one may distinguish between the aspects of number and space, while holding on to the primitive meaning of infinity in the literal quantitative sense of the word. Our number concept is dependent upon *primitive terms*, such as *discreteness* or *succession*, because the primitive meaning of the numerical aspect is found in *discrete quantity*. Natural numbers and integers, for example, are determined and delimited by arithmetical laws (operations), such as addition and subtraction, presupposing the numerical time order of succession, exemplified infinitely proceeding sequences of number. Our number concept therefore entails an endless succession, sometimes captured in the expression uncompleted infinity.

Eventually it became a practise to distinguish between uncompleted and completed infinity. And in 1831 Gauss protested: “So protestierte ich gegen den Gebrauch einer unendlichen Größe als einer vollendeten welches in der Mathematik niemals erlaubt ist” (quoted by Meschkowski, 1972:31). [“So I protested against the use of an infinite magnitude as completed, which is never permitted in mathematics.”]

One may therefore ask what prompted St. Augustine to call upon the actual infinite in his claim that what is outside the reach of the human intellect, namely the succession of the integers, could be overseen by God at once – as a completed magnitude.

The Bible distinguishes between the eternal Creator and temporal reality. The Bible itself explores various aspectual points of entry in a concept-transcending way. Consider the most basic biblical idea statement about God, namely that God is unique, there is but ONE God.

This idea of God’s uniqueness stretches the meaning of the numerical term *one* beyond the limits of the numerical aspect. For this reason, a concept of God is impossible, because concepts are based upon universal traits. If there would have been conditions for being a God then God would have been just one among many Gods. Yet the history of understanding God commenced with denying that God is infinite, with the exception of the *apeiron* of Anaximander – the unbounded-unlimited. From a historical perspective it is important to observe that whereas the other principles of origin, such as *air*, *water* and *fire*, were fluid in nature, the *apeiron* of Anaximander was *fixed* and *without aging*.

After St. Augustine advanced to the view that God can oversee all the natural numbers at once, a static element entered the scene, coupled with idea of eternity as the timeless present (a legacy of Plotinus – see *Enneads* III/7). This view continued to accompany the idea of actual infinity (the *at once infinite*). Wittgenstein still contemplates the two meanings of infinity when he remarks: “If we take eternity to mean not infinite temporal duration but timelessness, then eternal life belongs to those who live in the present” (Wittgenstein, 1961:64.311).

Contemplate a triangle without the spatial time order of simultaneity. Surely, its sides could not be given in succession. The term “triangle” literally speaks of a “three-in-one” – with a clear spatial connotation. Yet we can stretch the use of these *spatial terms* beyond the boundaries of the spatial aspect of creation, namely when we deduce from Scriptures the idea of the *triune* God [the Bible does not use this expression as such].

This idea of God’s uniqueness stretches the meaning of the numerical term *one* beyond the limits of the numerical aspect. For this reason, a concept of God is impossible, because concepts are based upon universal traits. If there would have been conditions for being a God then God would his uniqueness. Yet the history of understanding God commenced with denying that God is infinite, with the exception of the *apeiron* of Anaximander – the unbounded-unlimited. From a historical perspective it is important to observe that whereas the other principles of origin were fluid in nature, the *apeiron* of Anaximander was fixed and without aging.

13. Number and God’s identity

Contemplate a triangle without the spatial time order of simultaneity. Surely, its sides could not be given in succession. The term “triangle” literally speaks of a “three-in-one” – with a clear spatial connotation. Yet we can stretch the use of these *spatial terms* beyond the boundaries of the spatial aspect of creation, namely when we deduce from Scriptures the idea of the *triune* God [the Bible does not use this expression as such]. Doing this is similar to what we said above about ONE God.

Contemplate now the idea of identity. A “stretched” employment of our kinematic intuition provides us with the idea of *continued being* – which amounts in a concept-transcending use of the core kinematic meaning of *rectilinear* (*uniform*) *movement*. The first manifestation of such a concept-transcending use of the meaning of the kinematic aspect is given in our idea of *identity*. When we stretch this idea beyond creation we arrive at the idea of the *eternity* of God – compare Ex 3:14: *I am who I am*.

14. Lorenzen, Weyl, Cantor Tapp and Erasmus/Verhoef

Let us now return to the concept of *infinity*. The basic *concept* of infinity is given in the (purely arithmetical) understanding of *endlessness* – what one may call the *successive infinite* (SI). When infinity is turned inwards it provides us with the infinite divisibility of a continuum. Yet, as Lorenzen aptly remarks, arithmetic as such does not provide us with any motive to introduce the actual infinite. [In der Arithmetik – so muß man zusammenfassend sagen können – liegt kein Motiv zur Einfürhung von Aktual-Unenlichem vor" (Lorenzen, 1972:159). This includes what is presupposed in infinite divisibility, namely the spatial whole-parts relation. Russell concedes: "The relation of whole and part is, it would seem, an indefinable and ultimate relation" (Russell, 1956:138).

Weyl continues by pointing out that "Cusanus does not refer to the mystic form of passive contemplation, but rather to mathematics and its symbolic method … On the one side stands God as the infinite in perfection, on the other side man in his finiteness" (Weyl, 1932:8-9). He then recollects Aristotle's doctrine "that the infinite exists *dunamei*, in potentiality, in the state of becoming and ceasing to be, but not *energeia*". Unfortunately, according to Weyl, "the efforts to establish the foundation of analysis in the nineteenth century from Cauchy to Weierstrass, which start out from the limit notion, result in a new, powerful attempt to overcome the dynamics of the infinite in favor of static concepts: the theory of sets" (Weyl, 1932:71).

The basic sentiments of Weyl regarding infinity make it clear that his preferred choice is for a *dynamic* notion:

> For one who accepts this definition with its appeal to the infinite totality of numbers as having a meaning, the sequence of numbers open into infinity has transformed itself into a closed aggregate of objects existing in themselves, a realm of absolute existence which "is not of this world", and of which the eye of our consciousness perceives but reflected gleams (Weyl, 1932:72-73).

Interestingly, already in 1891 Husserl also had to limit himself to the *successive-infinite*. The unfortunate effect was that the follow-up volume of this work on the *Philosophy of Arithmetic* never appeared because Husserl could not develop the theory of the natural numbers without the use of the at once infinite. Lorenzen explains this with reference to an infinite decimal fraction: "als ob die unendlich vielen Ziffern alle auf einmal existierten". ("As if the infinitely many digits all exist at once" Lorenzen, 1972:163). Bernays explains – against Vaihinger with his philosophy of the *as if* – that he also uses the at once infinite in an *as if* sense, but that he does not mean anything internally contradictory (antinomic) with it (Bernays, 1976:60). Bernays thereby distances himself from the well-known "Philosophie des Als Ob" of Vaihinger (cf. Vaihinger, 1949:61-64).

For Weyl the continuum is not something static. With reference to Achilles and the tortoise he follows Aristotle by involving the successive infinite (Weyl, 1932:58). He highlights the "impossibility of conceiving the continuum as in a stage of rigid being" (Weyl, 1932:58) and on the next page affirms the view of Aristotle saying that the "moving does not move by counting" (Weyl, 1932:59).

Weyl now sets out to discover "the infinite in a form more primitive than that of the continuum, namely in the sequence of natural numbers 1, 2, 3, . . . and only with their help can we begin to attack the problem of the mathematical description of the continuum" (Weyl, 1932:61).

Nonetheless Weyl continues to wrestle with the difference between the successive infinite (SI) and the at once infinite (AI). We have quoted the phrase "infinite totality" from page 72 and find eleven pages further a remark about "the demand for totality and the metaphysical belief in reality inevitably compel the mind to represent the infinite as closed being by symbolical construction" (Weyl, 1932:83).

Weyl referred to "God as the completed infinite" (Weyl, 1932:84) and he called mathematics "the science of the infinite" (Weyl, 1932:7). But he continued the verdict of the dominant legacy since Aristotle by restricting infinity to the SI (successive infinite). However, throughout the history, whether or not defending the at once infinite (AI), the latter was associated with four decisive features: (i) the spatial order of simultaneity (of at once); (ii) the spatial whole-parts relation; and (iii) the idea of infinite totalities and (iv) the idea of a static quantum, firm and determined in all its parts – as Cantor defined it: "an actually infinite, in contrast, is a *Quantum* to be comprehended that on the one hand is *not variable*, but much rather in all its parts firm and determined, a genuine *constant*" (Cantor, 1962:401).

Cantor does not accept the potential infinite as genuinely infinite. Only when the at once infinite is present are we mathematically involved with infinity in the proper sense of the term. Cantor therefore distinguishes between "improper infinity", "proper infinity" (dealt with in set theory) and then God as the Absolute Infinite. Finally he explains that his theory does not only established the civil right (*Bürgerrecht*) of AI *sets* within the domain of mathematics but also as constructing the AI number concept (Cantor, 1962:411). Within the actual infinite he makes a threefold distinction: (i) the AI as the fully independent highest perfection realized *in Deo*; designated as the Absolute infinite or just the Absolute; (ii) in the second place insofar as it is realized in the dependent creaturely world; (iii) and in the third place insofar as it is mathematical “Größe, Zahl or Order Type” could be conceived *in Abstracto* (Meschkowski, 1967:111).

Tapp adds another angle to the reflection on the difference between the SI and the AI. His analysis could be compared with remarks made by Sinnige in his discussion of *Being* as it is understood by Parmenides. He argues that Parmenides described *being* in a twofold way, namely in terms with a *metaphysical* connotation and in *spatial terms* (Sinnige, 1968:86). Tapp points out that defining “the infinite as an ‘absolute’ does not help to overcome [these] fundamental difficulties, for here too one must distinguish between a *quantitative* and a *more comprehensive metaphysical meaning*” (Tapp, 2008:46).

> The ‘absolutely infinite’ in the quantitative sense [246-247] becomes only an almost vanishing part already in the real number level – how much less is an ‘Un-Menge’, like that of the ordinal numbers, in comparison to the metaphysical ‘everything’, that is to be encompassed by the Absolute (Tapp, 2008:246).
>
> [Die Bestimmung des Unendlichen als ein Absolutes’ hilft über diese prinzipiellen Schwierigkeiten nicht hinweg, denn auch hier muß man zwischen einer quantitativen und einer umfassenderen metaphysischen Bedeutung unterscheiden. Das ‘absolut -Unendliche’ im quantitativen Sinne wird [246-247] schon in der reellen Zahlenebene nur ein geradezu verschwindender Teil – wie viel weniger noch ist eine ‘Unmenge’, wie die der Ordinalzahlen, im Vergleich zum metaphysischen ‘Alles’, das vom Absoluten umfaßt werden soll.]

Tapp continues his argument by claiming that “mathematics and theology are not talking about the same subject when they speak of infinity: theology and metaphysics use infinity in a comprehensive sense, while in mathematics only the quantitative side is discussed. Within the limits set by this, there are mutual implications, for example, when one draws the metaphysical conclusion from set theory that the infinite can no longer be viewed simply as the indeterminate” (Tapp, 2008:246-247).

> [Mathematik und Theologie reden nicht über dasselbe Thema, wenn sie von Unendlichkeit sprechen: Theologie und Metaphysik benutzen Unendlichkeit in einem umfassenden Sinne, wahrend in der Mathematik nur die quantitative Seite thematisiert wird. Innerhalb der dadurch gesetzten Grenzen gibt es wechselseitige Implikationen zum Beispiel dann, wenn man aufgrund der Mengenlehre den metaphysischen Schluß zieht, daß das Unendliche nicht mehr einfach als das Unbestimmte angesehen werden kann.]

The last sentence of Tapp alludes to the dual nature of infinity, SI and AI. In 1925 Hilbert explained it as follows:

> Someone who wished to characterize briefly the new conception of the infinite which Cantor introduced might say that in analysis we deal with the infinitely large and the infinitely small only as limiting concepts, as something becoming, happening, i.e., with the potential infinite. But this is not the true infinite. We meet the true infinite when we regard the totality of numbers 1, 2, 3, 4, ... itself as a completed unity, or when we regard the points of an interval as a totality of things which exists all at once. This kind of infinity is known as actual infinity (Hilbert, 1925:167).

A different way to understand this is to ask whether or not the number 1 is equal to 0.999…? Suppose we accept only the SI, then there will always be increasingly smaller bits to be added. Alternatively, using the AI will allow us to define the number 1 as the ‘totality’ of the decimal expansion 0.999… We can explain this only by acknowledging that the original (and primitive) meaning of numerical succession imitates the *spatial time order of at once* as well as the fact that the parts of a spatial whole are also given at once. Russell holds that the “relation of whole and part” is “an indefinable and ultimate relation” (Russell, 1956:138). This spatial relation refers back to the SI manifesting a backward-pointing analogy of space to number. One may also observe the opposite direction in which any successive infinite multiplicity points forward to the spatial time order of at once, *as if* all the elements of a succession are given at once. One may call backward-pointing analogies *retrocipations* and forward-pointing analogies *anticipations*. Retrocipations are constitutive for the meaning of an aspect while anticipations need to be opened-up or disclosed under the guidance of theoretical thought. This distinction is manifest in the idea of the at once guided by the idea of infinite totalities, serving mathematics with a regulatively disclosed perspective. It should therefore also be kept in mind that terms from any aspect of reality could be employed in a conceptual and concept-transcending way.

This approach entails that *both* the SI and the AI could be employed in our concept-transcending understanding of God. Yet we have noted earlier that given the limitation regarding conceptional knowledge, restricted to universality, God is conceptually unknowable. Nonetheless there must be a form of knowledge in which we can *think* what is concept-transcending, namely a form of knowledge approximating what exceeds the grasp of conceptual knowledge. This is what could be designated as *idea-knowledge*. An idea in this technical sense may acknowledge the legitimate place of the at once infinite and infinite totalities within set theory.

Set theory should be seen as a spatially disclosed numerical theory opening up an *idea* of infinity. Bernays significantly explains this state of affairs. He writes, "Die Idee des Kontinuums ist eine geometrische Idee welche durch die Analysis in arithmetische Sprache Ausgedrückt wird" [The idea of the continuum is a geometrical idea expressed in the language of arithmetic (Bernays, 1976:74)] He continues on the same page with the remark that it is the *totality character* of the continuum that resists a complete arithmetization of the continuum."

15. Once more the nature of the infinite

Although modern mathematicians do not believe that the number of physical particles in the universe is actually infinite, Cantor nonetheless did hold this belief.

But when we want to attribute actual infinity to God, we cannot retreat to the substance concept with its distinction between essence and appearance – communicable and incommunicable properties. Infinity is supposed to be incommunicable.

A theo-ontological circle emerges if it is claimed that infinity is a Divine property from which mathematics receives its notion of infinity. However, it is rather the case that an exploration of the uniqueness and mutual relation between number and space serves as a point of orientation for our understanding of infinity in its two forms. But God neither is the successive infinite nor the actual infinite. God is not a number, not even when it is said there is but ONE God do we mean that God is the *number* one. Likewise, we should avoid holding the view that God is the completed infinite (Weyl) or the actual infinite (Cantor).

When we confess God's *omnipresence* it is understood to encompass the whole universe at once. Phrasing it in this way looks like a close approximation of the two forms of infinity in general distinguished, even though it avoids the dialectical opposition normally associated with these two options: eternity as *an endless time* and eternity as *timelessness*. We have argued that these two views merely explore the respective time orders within the numerical and spatial aspects: – *succession* and *simultaneity* (*at once*).

In their discussion of the *Kalām Cosmological Argument* (KCA) and the *Infinite God Objection*, Erasmus and Verhoef assign a key role to the idea of actual infinity. They circumscribe the potential infinite as a "a boundless quantitative process" and the actual infinite as "a boundless, completed totality of infinitely many distinct elements" "whose members are, nevertheless, present all at once" (The KCA "denies the existence of the actual infinite" – Erasmus & Verhoef, 2015)

16. Some concluding remarks

Discreteness entails succession (and progressions) and it underlies the most basic and primitive meaning of infinity: 1, 2, 3, … It is observed by imitating the spatial core feature of *wholeness* and the whole-parts relation. The primitive meaning of the whole-parts relation in turn brings to expression continuity, coherence, connectedness, totality – all of them mere synonyms of the core meaning of space and all of them reflect the irreducible meaning of spatial continuity. Given the primitive and irreducible nature of the totality character of continuity, it is incorrect to refer to the AI as the *completed* infinite because it would be a contradictory affirmation: how can the uncompleted infinite be *completed*? In subjection to the spatial time order of simultaneity every infinite totality is co-determined by this spatial time order of *at once*.

Once this meaning of infinity is turned inwards, evident in the infinite divisibility of a continuum, the inter-connection between space and number surfaces because the spatial concept of infinite divisibility points back to the numerical time order of the successive infinite. It is a retrocipation from the factual side of space to the law-side of the aspect of number. On the law-side the *number* of spatial dimensions reflects the foundational coherence with the numerical aspect. On the law-side of these two aspects the numerical order of succession anticipates the spatial time order of at once in the at once infinite. In other words, this anticipatory disclosure occurs under the guidance of the regulative hypothesis of the at once infinite which enables the mathematician to view any sequence of numbers as if it is given all at once.

The view opened up by Sinnige in his distinction between the use of certain terms in a spatial and in a metaphysical sense is similar to the distinction between quantitative and qualitative views introduced by Erasmus and Verhoef and also to the distinction drawn by Tapp between mathematical insights on the one hand and a comprehensive theological-metaphysical view on the other. We highlighted this distinction by introducing a closely related alternative distinction, namely that between conceptual-knowledge and concept-transcending knowledge.

The SI and AI are made possible by the law-order of creation which could be disclosed through human mathematical thinking when the numerical anticipations to space are theoretically deepened and disclosed. This view presupposes an acknowledgement of the *ontic givenness* of the modal aspects of reality.

A remarkable ambivalence in this regard is found in the view of Abraham Robinson. On the one hand, as an "inverse equivalent" of Cantor's transfinite numbers, he developed a non-standard analysis in which "infinitesimals" (numbers that are infinitely small) are used (Robinson, 1966:55 ff.), but on the other hand, he defends the belief that "infinite totalities do not exist in any sense of the word (that is, either really or ideally)" and then adds: "More precisely, any mention, or purported mention, of infinite totalities is, literally, meaningless." Nevertheless, he believes that mathematics should continue as normal: "That is, we should act as if infinite totalities really existed" (Robinson, 1979:507). Here, too, the successively infinite is implicitly used as a criterion to question the (disclosed modal) reality of infinite totalities. More recently Bell introduced his Smooth Infinitesimal Analysis (SIA).

While the non-standard analysis is founded on infinite totalities, SIA proceeds in a way that assigns priority to the continuous as "an autonomous notion, not explicable in terms of the discrete" (Bell, 2006:284).

What is needed is an understanding of the uniqueness and mutual coherence of number and space and the fact that the meaning of both these aspects (and their interrelations) could be employed (also theologically) in concept-transcending ways. But it should not treat God as a divine mathematician involved in mathematical operations with transfinite ordinal and cardinal numbers.

As founder of set theory, Cantor was convinced that "Set Theory deals with the actual infinite" (Robinson, 1966:39). By employing the idea of an infinite totality Cantor, in 1874, proved that the set of all real numbers cannot be enumerated in the same way as the set of all natural numbers, i.e., that the real numbers are *non-denumerable*. But exactly in this proof, which uses the at once infinite, Meschkowski (1972b:25) sees the "foundation of set theory".

Hilbert, arguably the greatest mathematician of the early 20th century, expresses the following view.

> The infinite has moved the human mind like no other question since the earliest times; the infinite has brought about mental stimulus and fruitfulness like virtually no other idea; the infinite however needs clarification like no other concept (Hilbert, 1925:163).

This article on actual infinity and God aims at highlighting some of the relevant distinctions required within this field.

Addendum

The underlying ontological perspective of this article is given in its non-reductionist approach, which is a cornerstone of Christian scholarship. This approach assumes the sovereignty of God and the liberating effect of properly distinguishing between God as Creator and creation. The principle of sphere-sovereignty prohibits the reduction of unique modal aspects to each other. It entails the perspective that the meaning of each aspect only comes to expression in its coherence with other aspects.

A critical appraisal of Dooyeweerd's views on the aspects of number and space is found in Strauss (2021). The issues at stake concerns the distinction between law-side and factual side, the numerical time-order of succession, the primitive meaning of the successive infinite, the factual whole-parts relation in the spatial aspect (which turns the successive infinite "inwards" – into infinite divisibility) and the regulative idea of infinite totalities. This provides us with the culmination point of our analysis in which the inter-modal coherence between number and space enables us to advance a unique understanding of the idea of actual infinity (the at once infinite). The distinction between the successive infinite and the at once infinite presupposes the unique Christian perspective on the irreducibility and mutual coherence between number and space. In this way it transcends the one-sided emphases found in the history of philosophy and mathematics, which either attempted to reduce space to number or number to space. From a Christian philosophical perspective it is possible to explore the alternative option mentioned above, namely to accept both the uniqueness and mutual coherence between number and space. These insights and distinctions are further enriched by the equally Biblically informed distinction between conceptual and concept-transcending knowledge. The way in which these issues are articulated is unique to our reformational tradition.

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