Reason: Its Kaleidoscopic Ideological Interface Part 3 – The Ontic Foundations of Rationality

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Samevatting

Benewens eensydige beklemtonings van die aard van die menslike 'rede' wat in die loop van die geskiedenis na vore getree het het dit tegelyk ook 'n inhoud ontvang wat swaar gelaai is met ontiese gegewens – tot so 'n mate dat daar aan die werklikheid self 'n rasionele karakter toegskryf is. Enersyds kan geen opvatting van die aard van rasionaliteit ontkom aan 'n diepste grondoortuiging wat rigting daaraan gee nie, en andersyds word aandag gevra vir die ontiese voorwaardes van rasionaliteit. Dit lei daartoe dat die beperkinge van die logika onderken moet word – soos tewens blyk uit die beginsel van toereikende grond wat die denke verwys na die ontiese gronde buite die logika. Hieraan moet die funderende rol van die beginsel van die uitgeslote antinomie toegevoeg word – wat tegelyk funderend is vir die logiese beginsel van non-kontradiksie. Die aard van die beginsel van die uitgeslote antinomie impliseer dat die menslike rede nie die 'koninklike' rol besit om in vermeende outonomie die werklikheid na willekeur te (re-) konstrueer nie, maar dat dit veel eerder die beskeie rol van 'dienskneg' moet aanneem – in diens van die blootlegging van ontspoorde denke wat in antinomieë vasloop. Vanuit hierdie gesigspunt word opnuut en in meer besonderhede ingegaan op Zeno se paradokse, op Kant se siening van antinomieë en op die antinomiese aard van die aritmetisiem in die wiskunde. Die problematiek van Kant se siening van rede-antinomieë word vervolgens behandel en die bespreking word afgesluit met 'n verwysing na die verhouding van Kant en Hilbert – waar laasgenoemde die gebruik van die opeens-oneindige wil regverdig met 'n beroep op die Kantiaanse verantwoording van die gebruik van 'rede-ideë' sonder om te besef dat Kant in sy argumentasie reeds die opeens-oneindige ingespan het.

1. The excessive load of rationality

In the two preceding articles the focus was mainly on the multiple forms and shapes that reason assumed in the course of the development of Western philosophy. These diverging faces of rationality may be a product of the apostate inclination of humanity to search within created reality for an ultimate principle of explanation, but the mere possibility to opt for the identity of thought and being or for peculiar kinds of reason, such as 'pure' reason, 'historical' reason, 'phenomenological' reason, 'hermeneutical' reason or 'foolish' reason, pre-supposes a given ontic diversity. As soon as we explore the implications of acknowledging this fundamental layer of rationality it will turn out that much of what is attributed to *reason* and *rationality* actually has a non-rational ontic character.

It has been pointed out that the thought-experiment of Galileo regarding a moving body inspired Immanuel Kant to explore the rationalistic leg of modern nominalism in his conviction that human understanding functions as the formal (*a priori*) law-giver of nature. Human understanding *prescribes* its laws to nature and does not derive them from nature (cf. Strauss, 2002:83; and Kant, 1783:320 = §36). The irrationalistic side of nominalism by contrast inspired historicism, the linguistic turn and postmodernism.

The idea that our world is 'rational' in the sense that it *conforms* to human reason expects too much from 'reason.' In fact, claims about *rationality* and its relation to *reality* ultimately reveal the *deepest convictions* of philosophers. One can designate this as the underlying commitment-dimension of reason and rationality.

2. The commitment-dimension of rationality

It is clear that already in Greek philosophy *rationality* never succeeded to disentangle itself from the deepest motivations of Greek culture and also constantly highlighted the structure of analysis as entailing identification and distinguishing. The argumentation found in Plato's dialogue *Parmenides* in a *negative way* underscores the *mutuality* (relatedness) of *identification* and *distinguishing*. And in the *Sophist* it is acknowledged that trying to know what *being* and *non-being* in themselves are present thought with an *aporia* (*ajporiva*), i.e., an unresolved problem.[1] In the Sophist 250e we read:

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1   Logic eventually used the term "aporia" in connection with the theoretical truth of a statement where there are grounds *for* and *against* it. In Latin *aporia* turned into *dubitation* and *question* (see Waldenfels, 1971:448).

So much then for a full statement of our problem. But since reality and unreality are wrapped alike in obscurity [aporia], there is for that reason some hope that any light, however much or little, cast on the one may likewise reveal the other. [251] And again, if we can get sight of neither, we will at least thus be able to investigate as best we can the logical relationship of both at once in some manner or other.[2]

But behind the correlation between *being* and *non-being* in Plato's thought the hidden *an-hypotheta* are the radical dualism of the *One* (*hen*) and the *Many* (*apeiron*) – which are not accessible to a logical (dialectical) method of investigation. Particularly in the dialogue *Parmenides* the *logical inaccessibility* of these opposing principles of origin is investigated – thus exhibiting the ultimate commitment of Greek thought to the underlying dualism between form and matter (the one and the many).

Plato argues that as soon as one tries to conceive the One in an absolute ("origin-al") sense, it escapes the grip of logical concept formation. The first and fourth paths of the dialectical argument which he developed in this dialogue (Parmenides) proceeds from the original Eleatic thesis that being in its absolute sense should be seen as a unity lacking plurality, motion, change and becoming. But then it cannot be said that it is a *whole*, for a whole is that which contains *all* its parts, which would imply that the One is *man*y (Parmenides, 137c4-d3). Likewise, the One is not *limitless* (137d7-8), and it is not *formless* (neither round, nor straight: 137d8-e1). Furthermore, the one is not anywhere (neither in itself, nor within something else), it cannot be in motion or be at rest, it cannot be identical or different from itself, and so on (138a-142a). Whatever is affirmed of the One will inevitably entail that it is fraud with plurality.

The fourth path pursues a similar line of argumentation with regard to the Many (the *apeiron* – see 159e-160b). The intermediate two paths demonstrate that all properties could be *affirmed* and *denied* in respect of both the *One* and the *Many* (142b-157b and 157b-159b)! The final conclusion to all four paths of this dialectical argument reads: "Therefore, if the One is, it is everything and nothing, in relation to itself and to the many" (160b1-3).

Similarly, we have seen that within the context of modern Humanism the idea of the logical (and eventual, historical, symbolical and social)

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2   This remark is made against the background of the supposed *opposition* of *rest* and *motion* (see Sophist, 250c and d). Being is supposed to transcend rest and motion.

construction of reality finds its ultimate root in the rationalistic and irrationalistic positions that are themselves in the grip of the ground motive of nature and freedom.

3. The structural issue: the ontic pre-conditions for rationality

Although we are used to speak about the *rational behaviour* of human beings, we never realize that the crux of this reference to rationality lies in something *more-than-rational* (or: something *pre-rational*).

Let us consider a few simple examples. If the directions given by a person about how to get from point A to point B in a big city are effective and optimal, we tend to say that the direction-giving was 'rational.' Likewise, a 'rational' person would not touch a live electrical cord bare-handed or jump from the fourteenth floor of a high building.

Now what is entailed in each one of these claims to *rationality*? In order to answer this question we should keep in mind that Paul Bernays, the co-worker of the foremost mathematician of the 20th century, David Hilbert, in his contribution to the Festschrift of Karl Popper, holds that *concept-formation* is decisive in an understanding of rationality.[3] But since concepts are always built upon *universal features* which are combined in the *unity* of a concept (as we have argued in the last section of the second article), the nature of universality should first of all be accounted for. The first remark to be made is that we should realize that our awareness of universality is made possible by the core meaning of space. Our intuition of space is bound up with extension and place, and in the case of universality with the notion of *everywhere*. Secondly we have to point out that *speaking* about universality – which is mediated by the meaning of space – does not exclude the reality of (spatial and non-spatial) universal features and relationships.

Our first example regarding the giving of directions in a large city requires a conceptual insight into the nature of *geo-spatial relationships*. Our rational understanding of spatial relationships does not *create* these relationships – at most one can say that the latter made possible our (spatial) concepts of direction and city-orientation. Knowing that it is life-endangering to touch a live electrical cord bare-handed or to jump from the fourteenth floor of a high building is also not *purely* 'rational.' Apart from the universal features of *physical* reality embodied in an electrical current and manifested by the *force* of gravity concept formation and rationality has no 'handle' on reality. The universal features of reality that

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3   He says that the "proper characteristic of rationality" is "to be found in the conceptual element" (Bernays, 1974: 601).

are *constitutive* for the acquisition of concepts are not *rational* – they form part of *ontic reality*. Yet, in most cases where we speak about the rational behaviour of human beings we tend to transpose these ontic features into the 'basket' of rationality.

In the context of the rationalistic legacy of modern philosophy this practice is understandable, for we have seen that the motive of logical construction leads to an identification of thought and reality (Hobbes, Kant, and Hegel). An amazing return to the rationalistic position of Kant and modernity is seen where Van Huyssteen identifies of the *structure* of the universe with *human rationality* and *mathematics*: "What is astounding, however, is to what extent our world is truly rational, i.e., in conformity with human reason" (Van Huyssteen, 1998:68). While mentioning Davies he refers to the "fact that the rational nature of our universe is reflected in its basic mathematical structure" (Van Huyssteen, 1998:71). Van Huyssteen and the modernist (rationalistic) tradition on this point do not distinguish between ontically given *universal features of reality* and the nature of concept-formation. From the fact that concepts are formed on the basis of universal traits it does not follow at all that these ontic properties themselves are *rational* in nature![4]

4. The limitations of logic

Although the logical principles of identity and non-contradiction enable the assessment of syllogistic arguments within traditional predicate logic they do not provide a *material* criterion of truth. The same applies to the application of these two principles in symbolic logic. Even when the principle of the excluded middle is added we are not making any headway, for in terms of these principles one can at most affirm that two contradictory statements cannot both be true at the same time and within the same context. This was already clearly understood by Kant. He says:

> Therefore the purely logical criterion of truth, namely, the agreement of knowledge with the general and formal laws of the understanding and reason, is no doubt a *condition sine qua non*, or a negative condition of all truth. But logic can go no further, and it has no test for discovering error with regard to the contents, and not the form, of a proposition (Kant, 1787-B:84-85).[5]

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4   This position is taken while at the same time an equally forceful attempt is made throughout the work to hold on to a postmodern perspective.

5   "Also ist das bloß logische Kriterium der Wahrheit, nämlich die Übereinstimmung einer Erkenntnis mit den allgemeinen und formalen Gesetzen des Verstandes und der Vernunft zwar die *conditio sine qua non,* mithin die negative Bedingung aller Wahrheit: weiter aber kann die Logik nicht gehen, und den Irrtum, der nicht die Form, sondern den Inhalt trifft, kann die Logik durch keinen Probierstein entdecken."

Only when an argument makes an *ontic* appeal, i.e., when some or other facet or feature of created reality is brought into the picture, is it possible to make truth claims. That this reference beyond the purely logical realm is required by a logical principle was noticed by Leibniz in his formulation of the logical principle of sufficient reason (ground). In 1813 Arthur Schopenhauer elaborated this insight in his dissertation. His preference was to call it the principle of *sufficient ground* of knowledge (*principium rationis sufficientis cognoscendi*).[6] The own formulation given to this principle by Leibniz is found in his Monadology:

> … and the second the *principle of sufficient reason,* by virtue of which we observe that there can be found no fact that is true or existent, or any true proposition, without there being a sufficient reason for its being so and not otherwise, although we cannot know these reasons in most cases (Leibniz, 1976:646 – see Sections 44 and 196).

If the nature of truth calls us to exceed the boundaries of logic, it also suggests that the ultimate criterion of truth will be ontical (in the sense of: more-than-logical) in nature. At once this will relativize the logical principle of non-contradiction because the underlying ontical principle requires that rational thinking should respect the given ontic diversity of reality in its irreducibility. The principle forbidding inter-modal *reductions* is known as the ontical principle of the *excluded antinomy* (*principium exclusae antinomiae* – see Dooyeweerd, 1997-II:36 ff.). This principle underlies the logical principle of contradiction and not the other way around. An illogical concept, such as a triangular circle, confuses two spatial figures – it violates the logical principle of (non-)contradiction. However, when two modal aspects of reality are confused, the violation of the principle of the *excluded antinomy* entails a logical contradiction but embodies an underlying *antinomy*, a clash of laws belonging to irreducible aspects. Zeno's classical arguments against multiplicity and movement demonstrates this distinction. Compare his fourth Fragment: "That which moves neither moves in the space it occupies, nor in the space it does not occupy" (Diels-Kranz B Fr.4). He first grants movement but then shows that motion is impossible – at least if one accepts his identification of something moving with 'successive'

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6   "As such it asserts that, if a judgement is to express a piece of knowledge, it must have sufficient ground or reason (*Grund*); by virtue of this quality, it then receives the predicate true. Truth is therefore the reference of a judgement to something different therefrom. This something is called the ground or reason of the judgement ..." (Schopenhauer, 1974:156). Already Plato knew that assertions require a foundation (see Timaeus 28a). Aristotle in turn distinguished amongst the *aijtivai* four causes: *material*, *formal*, *effective* and *final* ones.

'definite places' in space. Zeno reduces a moving entity to its *place*, thus eliminating its movement. This antinomy entails the following logical contradiction: something can move if and only if it cannot move. In other words, although every antinomy entails a logical contradiction, every logical contradiction does not necessarily presuppose an antinomy.

5. Rationality relegated from king to servant: restoring the primacy of ontic reality in its relation to rationality

The decisive turning points in the history of the various disciplines (both the natural sciences and the humanities) evince the powerful influence of on-going and constantly changing one-sided emphases, mostly captured in *ismic* orientations such as formalism, intuitionism and logicism (mathematics), determinism and indeterminism (physics), mechanism, vitalism and holism (biology), atomistic association psychology and holistic Gestalt psychology, etc. etc. All of these conflicting approaches proceed from specific (distorted) philosophical views of reality. Unveiling the inherent antinomies present in one-sided *isms* such as those mentioned constitutes the *service* that rationality ought to render to the scholarly endeavours of humankind.

5.1 Is 'force' an internally antinomic concept?

A first example of what we intend in this context may refer to the work of the German physicist Heinrich Hertz. In his *mechanistic reductionism* he only allowed for the employment of the basic concepts of *mass*, *space* and *time* in his understanding of the foundations of physics. Since he denied the original and primitive (irreducible) meaning of the *physical* aspect of the universe, he believed that the notion of "force" is internally antinomic (cf. Katscher, 1970:329). In the course of the 20th century it became clear that this mechanistic reduction cannot be maintained. Therefore it is understandable why Janich draws a clear distinction between *phoronomic* and *dynamic* statements. He states that the scope of the strict distinction between *phoronomic* (subsequently called *kinematic*) and *dynamic* arguments could be explained in terms of an example. Modern physics has to employ a *dynamic* interpretation of the statement that a body can alter its speed continuously only. Given certain conditions a body can never accelerate in a discontinuous way, that is to say, it cannot change its speed through an infinitely large acceleration, because that will require an infinite force.[7]

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7   "Die Tragweite einer strengen Unterscheidung phoronomischer (im folgenden kinematisch genannt) und dynamischer Argumente möchte ich an einem Beispiel

5.2 Another look at the paradoxes of Zeno

We commence by considering the approach of Withrow to the paradoxes of Zeno.

> It is not surprising that the *application* of the principle of the infinite divisibility of time is found to be associated with logical fictions formed, strictly speaking, in violation of the law of contradiction. For the principle itself involves just such a logical fiction [i.e., self-contradiction], as is evident when Zeno's Dichotomy paradox – which he appears to have formulated for a moving body – is applied to time itself ... before any interval can elapse a completed infinity of overlapping sub-intervals must have elapsed. One can, therefore, either conclude that the idea of the infinite divisibility of time must be rejected, or else if one wishes to make use of the device, one must recognize that it is, strictly speaking, a logical fiction [i.e., self-contradiction] (Withrow, 1961:152; see Grünbaum, 1967:65-66).

The accusation of introducing a (self-contradictory) "logical fiction" rests upon the idea of "a completed infinity." Two issues are involved in this account:

(i)   The possibility of the *infinite divisibility* of something continuous (Aristotle holds that "everything continuous is divisible into an infinite number of parts" – *Phys*. 238 a 22); and

(ii)   The idea of "a completed infinity."

In its own way the ambiguity present in the distinction between *endlessness* (the literal meaning of the infinite) and the idea of completed infinitude (an *infinite whole*) plays a role in the observations made by Kant about his first antinomy (*the world has a beginning in time, and is limited also with regard to space*). Kant also wrestles with the difference between what since Greek philosophy and mathematics is known as the *potential* and the *actual infinite* (Kant, 1787:B:458 ff.):

> I might have apparently proved my thesis too by putting forward, as is the habit of dogmatists, a *wrong* (Italics by the author – DFMS) definition of the

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erlautern, das ... aus der Protophysik stammt. Die Aussage 'ein Körper kann seine Geschwindigkeit nur stetig ändern' kann von der modernen Physik nur dynamisch verstanden werden. Geschwindigkeitänderungen sind Beschleunigung, d.h. als Zweite Ableitung des Weges nach der Zeit definiert. Zeit wird von der Physik als ein Parameter behandelt, an dessen Erzeugung durch eine Parametermaschine ("Uhr") de facto bestimmte Homogenitätserwartungen geknüpft sind ... Bezogen auf den Gang einer angeblich so ausgewählten Parametermaschine kann eine Körper seine Geschwindigkeit deshalb nich unstetig, d.h. mit unendlich große Beschleunigung änderen, weil dazu eine unendlich große Kraft erforderlich wäre" (Janich, 1975:68-69).

infinity of a given quantity. I might have said that the quantity is *infinite,* if no greater quantity (that is, greater than the number of given units contained in it) is possible. As no number is the greatest, because one or more units can always be added to it, I might have argued that an infinite given quantity, and therefore also an infinite world (infinite as regards both the past series of time and extension in space) is impossible, and therefore the world limited in space and time. I might have done this, but, in that case, my definition would not have agreed with the *true concept of an infinite whole* ("*einem unendlichen Ganzen*") (Italics by the author – DFMS). We do not represent by it how large it is, and the concept of it is not therefore the concept of a *maximum*, but we conceive by it its relation only to any possible unit, in regard to which it is greater than any number. According as this unit is either greater or smaller, the infinite would be greater or smaller, while infinity, consisting in the relation only to this given unit would always remain the same, although the absolute quantity of the whole would not be known by it. This, however, does not concern us at present.

Like Withrow also Kant (in this context) rejects the idea of "a completed infinity," of an *infinite totality* or *an infinite whole*. According to him "[T]he true transcendental concept of infinity is, that the successive synthesis of units in measuring a quantum, can never be completed" – to which he adds the following footnote: "This quantum contains therefore a multitude (of given units) which is greater than any number; this is the mathematical concept of the infinite."[8]

Rejecting the idea of completed infinitude is also used by Hermann Weyl in the context of his response to Zeno's antinomy of Achilles and the Tortoise. "The impossibility to comprehend the continuum in terms of static being cannot be formulated in a more pregnant way than through the known paradox of Zeno between Achilles and the tortoise" (1966:61). He mentions that the current solution of the paradox refers to the successive partial sums of the row $1/2 + 1/2^2 + 1/2^3 + \ldots$, $1 - 1/2^n$ ($n = 1, 2, 3, \ldots$) that does not grow beyond all limits (since they converge towards the number 1) and then adds the remark that when the infinitely many partial distances are viewed as a completed totality the essence of infinity is contradicted in the claim that Achilles in the end completely passed through the 'Unvollendbaren' (that which cannot be completed) (Weyl, 1966:61).[9]

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8   This note reminds me of the definition the professor in mathematics (A.P. Malan) formulated in response to my question in the first year class (1965): "what is the infinite"? He replied that it is a *number greater than any number*, *however large it my be*!

9   "Die Unmöglichkeit, das Kontinuum als ein starres Sein zu fassen, kann nicht prägnanter formuliert werden als durch das bekannte Paradoxon des *Zenon* von dem

This issue may be 'simplified' with reference to the number 1. The question is: is the number 1 equal to 0.999… or not? Suppose we only accept the potential infinite. Then there will always be more "fractional amounts" to be added, since however far one proceeds, there will always be more to come – in which case 0.999… is not equal to one. The other option is (in following Weiertrass, Dedekind and Cantor) to accept the "actual infinite" and straightaway to define the number 1 as the "totality" of the decimal expansion 0.999…!

Kaufmann and Fischer consider this account to be *internally antinomic*. The former says:

> It is generally accepted that an infinite decimal fraction indicates nothing else but a sequence of natural numbers, where ... the term 'sequence' does not bear upon an infinite totality, but merely pertains to the domain of a determined relationship (law-conformity) (Kaufmann, 1968:122-123).

Likewise, with reference to the square root of the number 2, Fischer argues:

> Every 'representation' of the square root of 2, whatever its nature may be, is only to be conceived of as an endless and absolute sequence of rational approximating values incapable of completion. Only when the self-contradictory fiction of completed infinitude is added, are we allowed to view the infinite decimal fraction as a representation of the square root of 2. Without the antinomic concept of 'completed infinity' we cannot form the concept of an irrational number (Fischer, 1933:108).

But what is entailed in the idea of "a completed infinity"? Although his constructivist approach also questions the idea of an infinite totality, Paul Lorenzen in a striking manner does explain its nature with reference to classical analysis (Weierstrass, Dedekind, Cantor). He explains that within this tradition "… each real number is represented as if the infinitely many decimal figures existed all at once" (Lorenzen, 1972:163).

This formulation actually captures a long-standing tradition which is closely related to the view of eternity as the *timeless present* (Plotinus

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> Wettlauf zwischen Achilleus und der Schildkröte. Der Hinweis darauf, daß die sukzessiven Partialsummen der Reihe $1/2 + 1/2^2 + 1/2^3 + …, 1 – 1/2n$ ($n = 1, 2, 3, ...$) nicht über alle Grenzen wachsen, sondern gegen 1 konvergieren, durch den man heute das Paradoxon zu erledigen meint, ist gewiß eine wichtige, zur Sache gehörige und aufklärende Bemerkung. Wenn aber die Strecke von der Länge 1 wirklich aus unendlich vielen Teilstrecken von der Länge 1/2 1/4 1/8, als 'abgehackten' Ganzen besteht, so widerstreitet es dem Wesen des Unendlichen, des 'Unvollendbaren', daß Achilleus sie alle schließlich durchlaufen hat."

Enneads, III/7) and that dates back even further to Parmenides (B Fr.8:3-6) where we find a characterization of *being* in terms of the *present* and of (a unified, coherent) spatial *wholeness*.

A way out of this apparent impasse is therefore given in the acknowledgement of the uniqueness and irreducibility of *space* as well as in an acknowledgement of the inter-modal connections between number and space. Whereas our basic awareness (intuition) of number is determined by the quantitative *order of succession*, that of space is determined by the order of *simultaneity* (lying at the basis of the kinematic order of *uniform flow*).

In Zeno's paradoxes Fraenkel discerns a tendency towards arithmticism: "… the tendency of arithmetization, already underlying Zenon's paradoxes, has been impressing its mark upon modern mathematics and may be perceived in axiomatic of set theory … as well as in metamathematics" (Fraenkel *et al.*, 1973:213). But he proceeds: "However, the converse direction is also conceivable, for intuition seems to comprehend the continuum *at once* (the author's italics – DFMS); mainly for this reason Greek mathematics and philosophy were inclined to consider continuity to be the simpler concept and to contemplate combinatorial concepts and facts from an analytic view" (Fraenkel *et al.*, 1973:213).

If the most basic meaning of infinity is expressed in an endless succession, or, as we may prefer to call it, in the *successive infinite*, then this quantitative (arithmetical) perspective retains its *relative* validity within a numerical framework – a context within which there is indeed no room for the idea of "actual" or "completed" infinity. But if the *totality character* of continuity is something original within the *spatial aspect*, then this feature of spatial continuity will stand in the way of the aim to arithmetize continuity completely. Paul Bernays realized that the mathematical sub-discipline known as "analysis" explores the interconnection between arithmetic and the meaning of spatial continuity: "The idea of the continuum is a geometrical idea which analysis expresses in terms of arithmetic" (Bernays, 1976:74).

This perspective opens the way for a *systematic account* of the meaning of the actual infinite, for the idea of an *infinite totality* points at a deepening and disclosure of the original primitive meaning of number: it entails that any successively infinite sequence of numbers could be viewed – in anticipation to the spatial feature of wholeness (totality) – *as if* all elements in the sequence are *at once* present as a whole, as an infinite totality!

The best way to capture this spatially deepened meaning of the (foundational) numerical (primitive) meaning of infinity is to designate it as the *at once infinite*. Phrased differently: through the idea of the at once infinite the meaning of number is deepened by pointing towards the original *spatial* meaning of wholeness / totality.

Since there is no *constructive* transition from the *successive infinite* to the *at once infinite* (cf. Wolff, 1971), it stands to reason that those who only accept the closed (not yet opened-up) meaning of number will *by definition* consider the *at once infinite* to be internally antinomic. But what *seems to be* an apparent antinomy turns out to be a meaningful configuration as soon as it is positioned within the context of a proper understanding of inter-modal coherences between number and space.

5.3 The antinomic nature of modern arithmeticism

Our preceding analysis finds support in the reaction of Paul Bernays[10] against the arithmeticistic inclination of modern mathematics since Weierstrass, Cantor and Dedekind. Compare his remark:

> The arithmetizing monism in mathematics is an arbitrary thesis. The claim that the field of investigation of mathematics purely emerges from the representation of number is not at all shown. Much rather, it is presumably the case that concepts such as a continuous curve and an area, and in particular the concepts used in topology, are not reducible to notions of number (Zahlvorstellungen) (Bernays, 1976:188).

Bernays based this conviction upon insights closely approximating what we have indicated above as the essentially *hypothetical character* of the opened up meaning of number:

> The position at which we have arrived in connection with the theory of the infinite may be seen as a kind of the philosophy of the 'as if'. Nevertheless, it distinguishes itself from the thus named philosophy of Vaihinger fundamentally by emphasizing the consistency and trustworthiness of this formation of ideas, where Vaihinger considered the demand for consistency as a prejudice ... (Bernays, 1976:60).

Although the deepened meaning of infinity is sometimes designated by the phrase *completed infinity*, this habit may be misleading. If *succession* and

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10   Bernays was the co-worker of David Hilbert – in 1934 and 1939 they published the two volume work: *Grundlagen der Mathematik* (*Foundations of Mathematics*) – and is known for a distinct axiomatization of set theory. He continued the publish actively until the 1970s.

simultaneity are mutually irreducible, then the idea of an infinite *totality* cannot simply be seen as the *completion* of an infinite succession. Therefore, when Dummett refers to the classical treatment of infinite structures "as if they could be completed and then surveyed in their totality," he (mistakenly) equates this "infinite totality" with "the entire output of an infinite process" (1978:56). The idea of an infinite totality simply transcends the concept of the successive infinite. Therefore 1 = 0.999… only when the decimal expansion is viewed as an infinite totality given at once.

On the basis of Cantor's set theory and subsequent developments in 20th century mathematics Grünbaum defend (what he believes to be) a "consistent conception of the extended linear continuum as an aggregate of unextended elements" (1952:288 ff.). He claims:

> The set-theoretical analysis of the various issues raised or suggested by Zeno's paradoxes of plurality has enabled me to give a *consistent* metrical account of an extended line segment as an aggregate of unextended points. Thus Zeno's mathematical paradoxes are avoided in the formal part of a geometry or chronometry built on Cantorean foundations. Given the aforementioned additivity rules for length of the standard mathematical theory, the consistency of the metrical analysis which I have given requires the *non-denumerability* of the infinite point-sets constituting the intervals on the line (1967:130).

Modern set theory indeed claims to define the 'continuum' purely in arithmetical terms, i.e., in terms of the actually infinite set of real numbers which is, due to Cantor's well-known diagonal proof, *non-denumerable*. Modern measure theory in addition uses the non-denumerability of the real numbers to transcend the zero measure of a denumerable set of points. Since addition is not defined on a non-denumerable set (if one cannot *enumerate* the elements of a set, one cannot *add* them), it looks as if a gap is found for the arithmeticistic claim that the uncountable set of real points could constitute a positive measure, larger than zero. Thus, so Cantor and the modern mathematical measure theory hold, a complete arithmetization of the 'continuum' is achieved.

Since there is simply no constructive way to bridge the gap between denumerable and non-denumarable infinity, Grünbaum had to admit that his entire argument crucially depends upon the non-denumerability of the real numbers: "The consistency of the metrical analysis which I have given depends crucially on the non-denumerability of the infinite point-sets constituting the intervals on the line" (1952:302).

However, the problem concealed in the non-denumerability of the real numbers is found in the fact that the proof of non-denumerability

presupposes the use of the at once infinite, while the idea of the at once infinite in turn presupposes the irreducibility of the totality-character of spatial continuity. Bernays is therefore completely justified in his remark: The property of being a totality "undeniably belongs to the geometric idea of the continuum. And it is this characteristic which resists a complete arithmetization of the continuum".[11] Since the at once infinite therefore presupposes the irreducible, unique nature of the spatial aspect it cannot be used subsequently to reduce space to number in terms of a non-denumerable set of real points. This attempt to reduce space to number is *antinomical* and it entails the following *contradiction*: space can be reduced to number if and only if it cannot be reduced to number![12]

6. Are Kant's 'antinomies' real antinomies?

The 'antinomies' formulated by Kant are: (1) "The world has a beginning in time, and is limited also with regard to space" versus "The world has no beginning and no limits in space, but is infinite, in respect both to time and space"; (2) "Every compound substance in the world consists of simple parts, and nothing exists anywhere but the simple, or what is composed of it" versus "No compound thing in the world consists of simple parts, and there exists nowhere in the world anything simple"; (3) "Causality, according to the laws of nature, is not the only causality from which all the phenomena of the world can be deduced. In order to account for these phenomena it is necessary also to admit another causality, that of freedom" versus "There is no freedom, but everything in the world takes place entirely according to the laws of nature" and (4) "There exists an absolutely necessary Being belonging to the world, either as a part or as a cause of it" versus "There nowhere exists an absolutely necessary Being, either within or without the world, as the cause of it" (see Kant, 1787:B-454-483).

The entire argumentation of Kant rests on the conviction that both the thesis and antithesis are conceivable without finding a contradiction within anyone of the two as such. It is only when they are juxtaposed that the yardstick of the logical principle of non-contradiction shows that they are (logically) incompatible in the sense that both cannot be true at the same time.

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11   "Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung des Kontinuums entgegensteht" – 1976:74.

12   That is, if and only if the at once infinite is used, which presupposes the irreducibility of the spatial aspect (wholeness/totality).

In his discussion of the thesis and antithesis of the first antinomy Kant argues against the at once infinite by applying the restricted (not-yet-opened-up) meaning of the successive infinite. In this case the at once infinite ("completed infinitude," an "infinite totality") is considered to be a self-contradictory idea, whereas in fact it is not the case if one acknowledges the interconnections between number and space. The 'solution' of the first antinomy takes recourse to 'endlessness' (the successive infinite) as well as to the distinction between the world of phenomena (*mundus phaenoumenon*) and the noumenal world (*mundus intelligibilis*).

Let us look at the second 'antinomy': "Every compound substance in the world consists of simple parts, and nothing exists anywhere but the simple, or what is composed of it" versus "No compound thing in the world consists of simple parts, and there exists nowhere in the world anything simple."

In his remarks about the first thesis of the second antinomy Kant states that space is not a *compositum* since in determining its parts space is a *totum*:

> We ought not to call space a compositum, but a totum, because in it its parts are possible only in the whole, and not the whole by its parts.

However, this proper insight into the nature of spatial wholeness (the totality-character of continuity) is burdened by the legacy of the Greek metaphysics of space, for Kant still (in line with Descartes) holds the view that matter is *qualified* by space, that it is essentially *extended* (cf. Kant, 1787:B35). This explains why he continues to speak about the *infinite divisibility* of matter (in his remark about the antithesis of the second antinomy) – without realizing that denying the physical nature (qualification) of material entities is truly *antinomic*. Whereas mathematical space – in a purely abstract and functional perspective – is both *continuous* and *infinitely divisible*, physical space is *neither* continuous nor infinitely divisible. Since it is bound to the quantum structure of energy physical space cannot be subdivided ad infinitum. Energy quanta indeed represent the limit of the divisibility of energy (see Hilbert, 1925:165).

In other words, although internally *antinomic* views do occur within the argumentation of Kant's first two antinomies (not recognized as such by him), the theses and antitheses of them are merely standing in a relation of *contradiction* to each other.

7. The "thing-in-itself" behind the problem of causality and freedom

By the end of the thirteenth and the beginning of the fourteenth centuries, the *nominalistic* movement questioned the realistic metaphysics of medieval scholasticism. Nominalism considers universals simply to be *substitutes*, referring in a signifying way to the *multiplicity of individual* things. It also introduces a new criterion of truth, restricted to what is present within the human mind. Truth concerns the *compatibility of concepts*. In emancipating the modern person from the authority of church belief (and the pope) early modern philosophy proclaimed its freedom and autonomy in a *Faustian* motive of control and domination that eventually manifested itself in the ideal of an all-encompassing natural scientific explanation in terms of cause and effect (causality). After Descartes this basic motivation explored different principles of explanation. Being acquainted with Galileo's law of inertia, Hobbes switched from *extension* as the essence of the matter in the thought of Descartes (the *res extensa*) to *moving body* as basic denominator. Leibniz tried to establish a balance between distinct monads and his principle of continuity (*lex continui*). Locke, Berkeley and Hume effectuated the transition from the initial mathematical science-ideal to a psygologistic one. But the deterministic consequences of the science-ideal threatened the ideal of human freedom, for if reality is entirely in the grip of physical causality, then the human person is also reduced to being fully determined without any freedom. Rousseau was the first philosopher who called modern philosophy back to a radical reflection on its truly deepest motivation: the Renaissance ideal of *free* and *autonomous* humanity**.**

> Nature commands every animal, and the brute obeys. The human being experiences the same impulse, but recognize his freedom to acquiesce or to resist; and particularly in the awareness of this freedom the spirituality of humankind manifests itself. ... but in the capacity to will, or much rather to choose, and the experience of this power, one encounters nothing but purely spiritual acts which are totally inexplicable through mechanical laws (Rousseau, 1975:47).

But it was Immanuel Kant who managed to subdue the science-ideal by restricting it to the categories of human understanding in their application to sense impressions only (ordered by time and space). This left open the super-sensory domain of practical reason where the autonomous personality can be free as an aim in itself (*Selbstzweck*). The apparently innocent distinction between appearance and thing-in-itself mediates Kant's demarcation of science-ideal and personality-ideal (*nature* and *freedom*). Consider his explanation:

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Now let us suppose that the distinction, which our Critique has shown to be necessary, between things as objects of experience and those same things as things in themselves, had not been made. In that case all things in general, as far as they are efficient causes, would be determined by the principle of causality, and conesquently by the mechanism of nature. I could not, therefore, without palpable contradiction, say of one and the same being, for instance the human soul, that its will is free and yet is subjected to natural necessity, that is, is not free. For I have taken the soul in both propositions *in one and the same sense*, namely as a thing in general, that is, as a thing in itself; and save by means of a preceding critique, could not have done otherwise. But if our Critique is not in error in teaching that the object is to be taken *in a twofold sense*, namely as appearance and as thing in itself; if the deduction of the concepts of understanding is valid, and the principle of causality therefore applies only to things taken in the former sense, namely, in so far as they are objects of experience – these same objects, taken in the other sense, not being subject to the principle – then there is no contradiction in supposing that one and the same will is, in the appearance, that is, in its visible acts, necessarily subject to the laws of nature, and so far *not free*, while yet, as belonging to a thing in itself, it is not subject to that law, and is therefore *free* (B, xxvii-xxviii).

To this we should add what he later on said in the CPR for it will show that his concern to safeguard the (autonomous) *freedom* of humankind necessitated this distinction between appearance and thing-in-itself. This is most evident from the entire *Transcendental Dialectic*. In his discussion of the solution of the third cosmological idea he once more explains that we are not allowed to ascribe any absolute reality to appearances: "The common but fallacious presupposition of the *absolute reality* of appearances here manifests its injurious influence, to the confounding reason. *For if appearances are things in themselves*, *freedom cannot be upheld*" (the author is italicizing in the last sentence – DFMS; 564; "Denn, sind Erscheinungen Dinge an sich selbst, so ist Freiheit nicht zu retten"). The final remark of this subsection reveals the basic motive of Kant's whole CPR: "My purpose has only been to point out that since the thoroughgoing connection of all appearances, in a context of nature, is an inexorable law, the inevitable consequence of obstinately insisting on the reality of appearances is to destroy all freedom. Those who thus follow the common view have never been able to reconcile *nature and freedom*" (the author is italicizing – DFMS; A,537, B,565).

Once it is understood that the ultimate tension between *nature* and *freedom* informed the distinction between appearance and thing-in-itself (*phenomena* and *noumena*), the fact that Kant resolved his antinomies by making an appeal to this latter distinction demonstrates to what an extent the basic motive of nature and freedom directed his thought.

8. Infinity and the ideas of reason: Kant and Hilbert

When Kant introduces the regulative use of his three reason-ideas – his ideas of the soul, the world and God – he explains that they are all to be used in a *regulative*, i.e., an "as if" way (cf. Kant, 1787-B:710-714). According to him any *constitutive* use, extending our understanding to new objects (cf. B, 20) only produces pseudo-rational dialectical concepts (the source of which Kant called the antinomies) (cf. B, 72). In relation to the problem of infinity, the striking aspect of his distinction is given in the fact that he actually used the (above discussed) idea of the *at once infinite* in order to explain the nature of his ideas of reason. To Kant the unconditioned is never to be met with in experience, but only in the *idea* – whenever "the conditioned is given, the entire sum of conditions, and consequently the absolutely unconditioned (through which alone the conditioned has been possible) is also given" (Kant, 1787-B:436). The expression "entire sum" (*ganze Summe*) clearly reveals the underlying (and transcendentally conditioning) role of the unique functional meaning of *space*.

What Kant rejected within the restricted domain of the application of the categories of understanding, namely the actual infinite (the idea of an infinite totality/whole), i.e., within the domain of the classical science-ideal restricted to the phenomenal world, is explicitly explored in his account of the regulative use of the ideas of reason.

However, David Hilbert did not realize that Kant actually structured his account of the nature of the ideas of reason on the basis of assuming the idea of the *at once infinite*. For otherwise he would not have called upon Kant's view of reason ideas in supprt of his defense of the employment of the infinite as a totality. Hilbert writes:

> The role that remains for the infinite to play is solely that of an idea – if one means by an idea, in Kant's terminology, a concept of reason which transcends all experience and which completes the concrete as a totality – that of an idea which we may unhesitatingly trust within the framework erected by our theory" (Hilbert, 1925:190 – translation from Benacerraf and Putnam, 1964:151).

The position of Hilbert is therefore circular: his justification for employing the at once infinite presupposes the at once infinite!

9. Concluding remark

Inherent in almost every *ism* found in the history of philosophy and the disciplines there is an accompanying elevation of human rationality. The reification (and deification) of human reason enthroned human rationality

to become autonomous and to reject any other (external) authority. Unfortunately, however, this over-estimation of reason got stuck in its own web, for scientific truth could only be rescued through the mediation of the logical principle of sufficient reason, pointing beyond logic to the ontic diversity within reality. And it turned out that this ontic diversity underlies the ontic principle of the excluded antinomy. Only when reason is 'positioned' within the context of logical norms, which themselves presuppose the said principle of the excluded antinomy, will it be possible to appreciate the role of 'servant' reserved for a modest understanding of reality within creation, because only then could reason be liberated from its reification to the role of 'king.'

Bibliography

BENACERRAF, H. & PUTNAM, P. 1964. *Philosophy of Mathematics*, *Selected Readings*. Oxford: Basil Blackwell.

BERNAYS, P. 1974. Concerning Rationality. In: *The Philosophy of Karl Popper*. *The Library of Living Philosophers*. Volume XIV, Book I, Schilpp, P.A. (Ed.). La Salle, Illinois: Open Court. pp.597-605.

DOOYEWEERD, H. 1997. *A New Critique of Theoretical Thought*. Collected Works of Herman Dooyeweerd. A-Series Vols. I-IV, General Editor Strauss, D.F.M. Lewiston: The Edwin Mellen Press.

DUMMETT, M.A.E. 1978. *Elements of Intuitionism*. Oxford: Clarendon Press.

FISCHER, L. 1933. Die Grundlagen der Philosophie und der Mathematik. Leipzig.

FRAENKEL, A., BAR-HILLEL, Y., LEVY, A. & VAN DALEN, D. 1973. *Foundations of Set Theory*. 2nd revised ed. Amsterdam: North-Holland Publishing Company.

GRÜNBAUM, A. 1952. Consistent Conception of the Extended Linear Continuum as an Aggregate of Unextended Elements. *Philosophy of Science*. Vol.19, nr.2. April. pp.288-306.

GRÜNBAUM, A. 1967. *Modern Science and Zeno's Paradoxes*. Middletown: Wesleyan University Press.

GRÜNBAUM, A. 1974. *Philosophical Problems of Space and Time*. Dordrecht (Holland): D. Reidel Publishing Company (2nd, enlarged ed.).

HILBERT, D. 1925. Über das Unendliche. *Mathematische Annalen*. Vol. 95. pp.162-190.

JANICH, P. 1975. Tragheitsgesetz und Inertialsysteem. In: *Frege und die moderne Grundlagenforschung*, Thiel, Chr. (Red.), Meisenheim am Glan.

KANT, I. 1787. *Kritik der reinen Vernunft* (1781); Felix Meiner, Ed. Hamburg 1956).

KATSCHER, F. 1970. Heinrich Hertz. In: *Die Grossen der Weltgeschichte*. Vol IX, Röntgen bis Churchill, München: R. Oldenbur.

KAUFMANN, F. 1968. *Das Unendliche in der Mathematik und seine Ausschaltung*. 2nd impression. Darmstadt: Wissenschaftliche Buchgesellschaft.

LORENZEN, P. 1972. Das Aktual-Unendliche in der Mathematik, In: Meschkowski, H. (Ed.), Grundlagen der Mathematik. Darmstadt: Wissenschaftliche Buchgesellschaft. pp.157-165.

ROUSSEAU, J.J. 1975. *Du contrat social et autres oeuvres politiques*. Paris: Editions Garnier Fréres.

SCHOPENHAUER, A. 1974: *On the Fourfold Root of the Principle of Sufficient Reason*. Translation by Payne, E.F.J. La Salle, Ill.: Open Court.

STRAUSS, D.F.M. 2002. Reason: Its Kaleidoscopic ideological interface. *Journal for Christian Scholarship*. 38(Special Edition):1-12.

WALDENFELS, B. 1971. *Aporie*, *Aporetik*. In: Ritter. pp.447-448.

WHITROW, G. J. 1961. *The Natural Philosophy of Time.* London: Thomas Nelson & Sons Ltd.

WEYL, H. 1966. *Philosophie der Mathematik und Naturwissenschaft*. 3rd revised & expanded edition. Vienna.