Is a Straight Line the Shortest Distance between two Points and does 2+2 equal 4?

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Aritmetiese optelling (2+2=4) en 'n geval van geometriese optelling (die vektorsom $\underline{2}+\underline{2}=\sqrt{\underline{8}}$_) dui op verskillende soorte feite. Hierdie feite is nie 'brute' feite nie, aangesien hulle bepaal word deur ooreenstemmende getalswette en ruimte-wette. Die oorspronklike ruimtelike aard van 'n lyn (soos gegee in die een-dimensionele uitgebreidheid daarvan) kan nie bloot gedefinieer word deur die maat van die uitgebreidheid daarvan nie, waar hierdie uitgebreidheid uitdrukking in die nosie van afstand (al dan nie, in die sin van 'n metriese ruimte verstaan) vind. Laasgenoemde kan slegs gespesifiseer word deur 'n getalswaarde aan die lengte van 'n lyn toe te ken – maar 'n lyn self is sekerlik nie 'n getal nie. Die blote onderskeiding tussen verskillende ruimtelike dimensies (een, twee en meer dimensies) vereis 'n implisiete verwysing na getal (die getalle 1, 2, ensomeer). Sodra die onderskeidenheid van getal en ruimte onderken word ontstaan die vraag watter een van hierdie twee gebiede meer basies is (in die sin dat dit die sin van die ander aspek veronderstel) en hoe die onverbreeklike samehang tussen beide aspekte daar uitsien. Teen die einde word_

aangetoon hoedat die drie primitiewe terme wat in Hilbert se aksiomatisering van die meetkunde gebruik word 'n uitdrukking is van die ruimtelike subjek-objek-relasie.

1. Law and factuality

The statement that "a straight is line the shortest distance between two points" seems to be as self-evident as the statement that "2 + 2 = 4". In an earlier phase of his development Russell 'corrected' this definition: "A straight line, then, is not the shortest distance, but is simply the distance between two points" (Russell, 1897:18). The three key terms in this statement concern *spatial* configurations, namely the terms 'line', 'point' and 'shortest'. Yet the key element maintained in Russell's improved definition echoes something of our awareness of numerical relations: *distance*.[1] If this is indeed the case it may turn out that an analysis of this statement will at once get entangled in the consideration of arithmetical and spatial issues, which means that it cannot be analyzed purely in spatial (or geometrical) terms. In order to broaden our perspective such that considerations stemming both from the domains of number and space will receive their due attention, we introduce another 'sum' – one in which it is alleged that 2+2 is not equal to 4 but to √8.

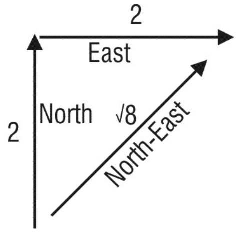
At first sight this alternative sum, 2+2=√8, may be questioned, owing to the fairly widespread conviction that mathematical issues are instances of a compelling rationality. Recently Fern for example wrote:

> Mathematical calculations are paradigmatic instances of universally accessible, rationally compelling argument. Anyone who fails to see "two plus two equals four" denies the Pythagorean Theorem, or dismisses as nonsense the esoterics of infinitesimal calculus forfeits the crown of rationality (Fern, 2002:96-97).

Someone who shares the conviction of Fern may want to reinforce the original claim, namely that 2+2=4, by referring to the addition of 2 fingers and another two 2 fingers, which indeed adds up to 4 fingers. Apparently this specified addition conclusively confirms the soundness of the initial statement that 2+2 is equal to 4. Unfortunately the issue is more complicated than it may seem at first sight, for the alternative assertion, namely that 2+2=√8, implicitly changed the context of addition, for when a person walks 2 miles north and afterwards 2 miles east, then that person

1 We restrict our discussion to *metrical* spaces. Mac Lane accepts space as "something extended" and on the basis of the notion of 'distance' defines a *metric space* (see Mac Lane, 1986:16-17). An explanation of the mutual relation between discreteness and continuity within a *topological* context requires a different argument. A starting-point for such a discussion is found in White (1988:1-12).

will be √8 miles away from the initial point of departure. This context concerns *spatial addition*, that is mathematically treated in *vector analysis*, where a vector possesses both *distance* (magnitude) and *direction*.[2] One can capture this altered context by underscoring the numerals involved in order to specify the fact that we are dealing with vectors: $\underline{2}+\underline{2}=\sqrt{\underline{8}}$. The upshot is that we now clearly have two different *kinds of facts* related to addition at hand: a *numerical* fact (designated as 2+2=4) and a *geometrical* fact (designated as $\underline{2}+\underline{2}=\sqrt{\underline{8}}$). In order to capture the specifications of this example one may construct the following figure:



These facts are not unqualified – that is to say, they are distinct because they are differently qualified, respectively as *numerical* and as *spatial*. They are therefore not simply 'facts' in themselves. In their factuality they are *delimited* by alternative order-determinations. The operation of numerical addition displays an order-determination different from the operation of spatial addition, as is clearly manifested in the alternative sums: 4 and $\sqrt{\underline{8}}$. In our example the underlying "order diversity" therefore makes possible the indicated distinction between numerical and spatial facts.

But there is something else present in this distinction between two kinds of facts, namely the reference to the operation of addition. Modern mathematical set theory normally first of all approaches this domain in terms of the algebraic structure of *fields* – where the (binary) operations called addition (+) and multiplication (.) meet the field axioms (specified as *laws*).[3] The fact that addition and multiplication within a system of

2 In the first half of the 19th century Grassmann already introduced the idea of a vector. He designnated such a line segment ('Strecke') with a specific direction and length on a specific straight line as a "linienteil" which became more generally known in German literature as a *vector* ('Vektor'): "Graßmann nannte eine solche Strecke bestimmter Richtung und Länge auf einer bestimmten Geraden einen *Linienteil*; jetzt ist in der Deutschen Literatur der Name Vektor üblicher" (Klein, 1925:24). Hedrick and Noble mistranslated "Richtung und Länge" as "length and sense," perhaps because later on in the same original German paragraph Klein himself used the German words "Länge und Sinn") – also explaining why the word order of "Richtung und Länge" was reversed to "length and sense" by them (see Klein, 1939:22).

3 A *field* is defined as a set *F* such that for every pair of elements *a*, *b* the sum *a*+*b* and the product *ab* are still elements of *F* subject to the associative and commutative laws

numbers yield numbers belonging to the initial set is also mathematically articulated by saying that the *system of numbers* under consideration is *closed* under the operations (laws) of *addition* and *multiplication*. The most basic instance of this strict correlation between (arithmetical) laws and numbers subject to them is found in the system of natural numbers where it is immediately evident that the addition and multiplication of any two natural numbers once more yield natural numbers (s = system; t = set):

system of na-	operations / laws:	(+,x)
ural numbers N_s	numerical subjects:	$N_t = (1, 2, 3, ...)$

The designation 'system' therefore comprises both arithmetical laws and arithmetical subjects – in the sense that the laws (operations) not only *determine* the behavior of the subjects but also *delimit* them. What has been explained above therefore means that the system of natural numbers finds its *determination* and *delimitation* in the operations of addition and multiplication that are closed over the set of natural numbers – in the sense that adding or multiplying any two natural numbers will always yield another natural number.[4]

Introducing further arithmetical laws or operations will invariably call for additional (correlated) numbers that are factually subjected to these new determining and delimiting arithmetical laws. For example, if the operation of subtraction is added to those of addition and multiplication, the correlating set of integers (Z_t) is constituted – and considered in their correlation this yields the *system of integers*. Likewise, extending the arithmetical operations by introducing *division* the correlating set of fractions is needed within the system of rational numbers.

system of ratio-	operations / laws:	(+,x, –, ÷)
nal numbers Q_s	numerical subjects:	$Q_t = (a/b; a,b \in Z_t / b \neq 0)$[5]

for addition and multiplication, and combined to the presence of a *zero element* and a *unit* (or *identity*) *element* (see Bartle, 1964:28; Berberian, 1994:1 ff.). This definition of a field is then expanded to that of an ordered field and it is finally combined with the idea of completeness.

4 The ultimate presupposition of these operations is found in the numerical order of succession. The latter is *primitive* and comes to expression in the principle of induction which, according to Weyl, safeguards mathematics from collapsing into an enormous tautology (1966:86). The Peano axioms (for the positive integers) yield a mathematical articulation of this primitive arithmetical order of succession. The correlation of the operations of addition and multiplication and their delimiting and determining role in respect of numerical subjects are consistent with Peano's axioms because they are entailed in the *complete ordered field of real numbers* (see Berberian, 1994:230).

5 This explanation, in terms of the strict correlation between operations at the law side and numerical subjects at the factual side, is *formally* similar to the way in which Klein introduces negative numbers and fractions (by means of the reverse operations of addition and multiplication – see Klein, 1932:23 ff. & 29 ff.).

Against this background it is clear that the systematic arithmetical statement 2+2=4 does not designate a "brute fact" (a fact "in itself," "an sich"), since the factual relation specified for numerical subjects (selected from the set of natural numbers) that are involved in it, exhibits the *measure* of the numerical law of addition. One can also say that this statement conforms to the determining and delimiting effect of the arithmetical law of addition. Consequently, the statement that 2+2 is equal to 4 concerns a law-conformative (arithmetical) state of affairs – it displays a specific lawfulness or orderliness for it meets the conditions set by the presupposed arithmetical order.

Envisaging all arithmetical laws at once suggests that we may speak of a unique *sphere of laws* inextricably correlated with diverse sets of numbers subjected to these laws. Another way to capture this situation is to speak about a numerical sphere in which arithmetical laws are strictly correlated with arithmetical subjects (numbers); in other words within this numerical domain a distinction is made between its *law side* (order side) and its *factual side*. Myhill, who appreciates Brouwer as the originator of "constructive mathematics," introduces the notion of a 'rule' (the equivalent of what we have designnated as "law side") as "a primitive one in constructive mathematics"; "We therefore take the notion of a rule as an undefined one" (Myhill, 1972:748).[6] In his encompassing introduction to set theory (the third impression), Adolf Fraenkel refers to the peculiar *constructive* definition of a set which accepts as a foundation the *concept of law* and the *concept of natural number* as *intuitively given*.[7]

The geometrical sum: $\underline{2}+\underline{2}=\sqrt{\underline{8}}$ belongs to a different domain, to a different sphere of laws, one where it is also possible to distinguish between a law side (order side) and a factual side. The sphere of spatial laws differs from the sphere of numerical laws – in an exemplary way expressed in the difference between 2+2=4 and $\underline{2}+\underline{2}=\sqrt{\underline{8}}$.

2. Distance

We may now return to the mentioned key element in the modified definition given by Russell, *distance*: a line "is simply the *distance* between two points".

The first observation to be made in this connection is to establish that the notion of a 'line' as the 'distance' between two 'points' concerns *spatial*

6 Myhill received his Harvard Ph.D. under W.V. Quine.

7 "Ohne die Stellung wieterer intuitionistischer Gruppen und anderer Richtungen … zum Mengenbegriff zu schildern, sei hier noch auf die wesentlich abweichende Auffassung Brouwers hingewiesen. Dieser stellt eine eigenartige rein *konstruktive* Mengendefinition an der Spitze, bei der der Begriff der *natürliche Zahl* und der des *Gesetzes* als intuitiv gegeben zugrunde gelegt werden" (Fraenkel, 1928:237).

realities. A line is a spatial configuration, not an arithmetical one. Yet the crucial question is: how can one designate the 'distance' between two points? The answer is: by specifying a *number* (for example by saying it is 3 inches long). The problem with this answer is that something *spatial*, namely a 'line', is now apparently equated with something *numerical*, namely 'distance'![8] Does this mean that the domains of space and number are coinciding? If it is the case, then a question of priority arises: is space numerical (then a 'line' is identical to 'distance', i.e., to number), or is number spatial (then number, i.e., 'distance' is identical to space, i.e., a 'line')?[9] The situation is further complicated by the fact that the number specified (such as '3') does not stand on its own, i.e., it appears within a non-numerical context – one in which the general issue of *magnitude* prevails, with *length* as a one-dimensional magnitude. And to add insult to injury, we now suddenly have to account for another spatial notion: *dimensionality*! But still new problematic questions are generated, for in our example of "3 inches" – related to the extension of a line – the reference to length brought with it the (spatial) perspective of *one* dimension (length specifies magnitude in the sense of one dimensional extension). On the one hand this suggests *extension*, which presumably essentially belongs to our awareness of space, while at the same time, just as in the case of the term 'distance', it reveals a connection with number, for one can speak about 1-dimensional extension (magnitude; i.e., of length), 2-dimensional extension (magnitude; i.e., of area), 3-dimensional extension (magnitude; i.e., of volume), and so on. Even if priority is given to the spatial context by admitting that the distinction between different dimensions is indeed something spatial, no one can deny that in some or other way number here plays a foundational role, for without number the given specification regarding 1, 2, or 3 dimensions is unthinkable.

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8 In passing we note that the term 'distance' in yet a different way evinces an intrinsic connection with the meaning of number because a line is supposed to be the distance between **two** points. Multiplicity is numerical; a multiplicity of *points* is spatial. Furthermore, the term 'inch' here has the function of the *unit of measurement*, i.e. the unit length. Therefore this unit is on a par with the notion of distance, because number 1 and the number 3 respectively represent these two lengths.

9 Of course this concise dilemma reflects the basic contours of the history of mathematics as a discipline, for the initial Pythagorean claim was that everything is number. With the discovery of irrational numbers (where it turned out that there are 'incommensurable' ratios) mathematics was turned into geometry. However, during the 19th century – owing to the work of Cauchy, Weierstrass, Dedekind (1887, 1901) and Cantor(1962) arithmeticism once more gained the upper hand, although it should be remembered that Frege, after the failure of his logicist program, close to the end of his life, reverted to the view that mathematics essentially is geometry: "The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis – a geometrical one in fact – so that mathematics in its entirety is really geometry" (Frege, 1979: 277).

Clearly, the term ‘distance’ is embedded within the domain of space and it also evinces a strict correlation between an *order of extension* (the law side of this domain – i.e., dimensionality) and *factually extended spatial subjects* – spatial figures (such as 1-dimensional ones, i.e, lines), 2-dimensional ones, i.e., areas) and 3-dimensional ones (i.e., volumes).

The complexities generated by a consideration of extension in the sense of an order of extension and of factually extended spatial subjects (spatial figures) adds weight to the suggestion that, although something like a line has a spatial nature, its extended character cannot reveal its true spatial meaning without showing a dependence upon the meaning of number. The reason for this acknowledgement is found in the intrinsic role of numerical terms that are ‘coloured’ by space, such as *distance* and *dimension*. Within a numerical context, such as what is mathematically known as “real analysis,” one can easily dispense with the concept of distance. But textbooks on real analysis sometimes still acknowledge that the geometric meaning of the term ‘distance’ may be useful, for “instead of saying that $|a – b|$ is ‘small’ we have the option of saying that *a* is ‘near’ *b*; instead of saying that ‘$|a – b|$ becomes arbitrarily small’ we can say that ‘*a* approaches *b*’, etc.” (Berberian, 1994:31)

3. Back to space

Two years after Russell gave his mentioned modified definition of a line as the distance between two points, the German mathematician, David Hilbert, published his axiomatic foundation of geometry: *Grundlagen der Geometrie* (1899). In this work Hilbert abstracts from the contents of his axioms, based upon three *undefined* terms: “point,” “lies on,” and “line.” Suddenly the term ‘distance’ disappeared. The next year, when Hilbert attended the second international mathematical conference in Paris, he presented his famous 23 mathematical problems that co-directed the development of mathematics during the 20th century in a significant way – and in problem 4 he provides a formulation that opens up a new perspective on this issue, for instead of speaking of the *distance* between two points he talks of a straight line as the (shortest) *connection* of two points.[10] This choice of words completely avoids the traditional view, even found in the work of a contemporary mathematician who still believes that the “straight line is the shortest distance between two points” (Mac Lane, 1986:17).

Hilbert’s German term ‘Verbindung’ (‘connection’) does not define a line since it presupposes the meaning of continuous extension. Every part of a

10 “[Das] Problem von der Geraden als kürzester Verbindung zweier Punkte” (see Hilbert, 1970:302).

continuous line coheres with every adjacent part in the sense of being connected to it. Although it is tautological to say that the parts of a continuous line are fitted into a gapless coherence, it says nothing more than to affirm that the parts are connected. In this sense the connection of two distinct spatial points also highlights the presence of (continuous) spatial extension between the points that are connected to each other. In other words, two points cannot be connected by a third point, but only by means of a line, i.e. through *spatial extension*.

Combined with the primitive terms employed in his axiomatic foundation of geometry (‘line’, ‘lies on’ and ‘point’) the term ‘connection’ no longer equates a line with its distance. Once ‘liberated’ from this problematic bondage, alternative options emerge in order to account for the meaning of the term ‘distance’. If *distance* is the 1-dimensional *measure* of factual (spatial) extension, then one can do two things at once:

(i) acknowledge the spatial context of this measure (1-dimensional magnitude) and

(ii) allow for the reference to number that is evident both in the ‘1’ of 1-dimensional extension and in the (numerically specified) *length* evident in ‘distance’ as a specified (factual) spatial magnitude.

The core meaning of space, related to the awareness of extension and dimensionality, now acquires a new appreciation, further supported by the undefined nature of the term ‘line’ in Hilbert’s 1899 work. The message is clear: if the core meaning of space (extension) is indefinable and primitive, then it is impossible to attempt to *define* a line by using a non-original term within space, such as the term ‘distance’. Distance as the measure of extension of a (straight) line depends upon and presupposes the existence of the line in its primitive 1-dimensional extension and can therefore never serve as a definition of it. Therefore the ‘definition’ of a (straight) line as “the distance between two points” (Russell) presupposes what it wants to define and consequently begs the question.

3.1 What is presupposed in space?

In our discussion of the question whether or not the domains of space and number are coinciding we have started by analyzing some consequences of the option that they do coincide. However, this led to an acknowledgement of the fact that every specification of spatial configurations is unavoidably connected with terms reflecting some or other coherence with number (magnitudes and the number of dimensions). This outcome opens up the way to an alternative: investigate the consequences of the assumption that although space and number are unique and distinct they still unbreakably cohere. The new question to be analyzed is then: what is the interrelation between the spatial and the numerical?

If the measure of the factual (one dimensional) extension of a straight line could be specified by its distance, then the distance of a line not only presupposes its spatial extension since it also presupposes the intrinsic interconnection between the meaning of space and the meaning of number. Various mathematicians had an appreciation of this state of affairs. Paul Bernays (the co-worker of Hilbert), for example, says that the idea of continuity is a geometrical idea which is expressed by analysis in an arithmetical language (Bernays, 1976:74).[11]

But let us consider further options. The mere possibility to juxtapose two distinct 'facts', such as the statements that 2+2=4 and $\underline{2}+\underline{2}=\sqrt{\underline{8}}$, points in the direction of acknowledging two unique domains – each with its own sphere of laws and correlated subjects. But this basic acknowledgement does not solve the subsequent problems, for the following two issues are still in need of clarification:

(i) which one of these two domains is more fundamental, in the sense of *foundational*, to the other? and
(ii) how should one account for the interconnections (interrelations) between these two domains?

3.1.1 Which region is more basic?

Let us start with the approach of Bernays where he considers the way in which one can distinguish between our *arithmetical* and *geometrical* intuition. He rejects the widespread view that this distinction concerns *time* and *space*, for according to him the proper distinction needed is that between the *discrete* and the *continuous*.[12] But then the question recurs: what is the relationship between the 'discrete' and 'continuous'?[13]

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11 "Die Idee des Kontinuums ist eine geometrische Idee, welche durch die Analysis in arithmetischer Sprache ausgedrückt wird."

12 "Es empfiehlt sich, die Unterscheidung von 'arithmetischer' und 'geometrischer' Anschauung nicht nach den Momenten des Räumlichen und Zeitlichen, sondern im Hinblick auf den Unterschied des Diskreten und Kontinuierlichen vorzunehmen" (Bernays, 1976:81). Rucker also states: "The discrete and continuous represent fundamentally different aspects of the mathematical universe" (Rucker, 1982:243).

13 The problem concerning which one is more basic – number or space – cannot be solved by a *genus proximum* – albeit that of Aristotle with his distinction between a discrete quantity and a continuous quantity or that of the structuralist Resnik with his distinction between discrete patterns and continuous patterns (cf Aristotle: "Quantity is either discrete, or continuous" – *Categ*.4 b 20; and Resnik, 1997:201 ff. 224 ff.). In terms of the distinction between the domain of number and that of space the term "pattern" in the first place derives its meaning from *spatial configurations* or *patterns*. Only afterwards one can stretch this term – metaphorically or otherwise – in order to account for quantitative relations as well. Whatever the case may be, speaking of "discrete patterns" just as little bridge the gap between discreteness and continuity as referring to the "domain of number" does it (where the term "domain" also derives

Fraenkel *et.al*. even speak of a 'gap' in this regard and add that it has remained an "eternal spot of resistance and at the same time of overwhelming scientific importance in mathematics, philosophy, and even physics" (Fraenkel *et.al*., 1973:213). These authors furthermore point out that it is not obvious which one of these two regions – "so heterogeneous in their structures and in the appropriate methods of exploring" – should be taken as starting-point. Whereas the "discrete admits an easier access to logical analysis" (explaining according to them why "the tendency of arithmetization, already underlying Zenon's paradoxes may be perceived in [the] axiomatics of set theory"), the converse direction is also conceivable, "for intuition seems to comprehend the continuum at once," and "mainly for this reason Greek mathematics and philosophy were inclined to consider continuity to be the simpler concept" (Fraenkel *et.al*., 1973: 213). Of course the modern tendency towards an arithmetized approach (particularly since the beginning of the 19th century) chose the alternative option by contemplating the primary role of number. Although Frege – as mentioned in note three above – by the end of his life equated mathematics with geometry (consistent with the just mentioned position of Greek mathematics), his initial inclination certainly was to opt for the foundational position of number. Already in 1884 he asked if it is not the case that the basis of arithmetics is deeper than all our experiential knowledge and even deeper than that of geometry?[14]

From our discussion of the difference between an arithmetical and a spatial sum and in particular from our remarks about the term 'distance' it may be possible to derive an alternative view on the order relation between the regions of discreteness and continuity. Suppose we consider the idea that *discreteness* constitutes the core meaning of the domain of number and that *continuous extension* highlights the core meaning of space. Then these core meanings guarantee the distinctness (uniqueness) of each domain. The domain of number, with its sphere of arithmetical laws and numerical subjects, is then seen as being stamped or qualified by this core meaning of *discreteness*. Likewise the domain of space, with its sphere of spatial laws and spatial subjects, is then viewed as being qualified by the core meaning of *continuous extension*.

But we have seen that a basic spatial subject, such as a (straight) line, cannot be understood without some or other reference to the meaning of number, for observing the *measure* of the line's extension requires the

from the meaning of space). The issue at stake in this connection is one falling outside the scope of this article for it concerns what should be treated in an analysis of the *elementary basic concepts* of a scholarly discipline (such as mathematics).

14 "Liegt nicht der Grund der Arithmetik tiefer als der alles Erfahrungswissens, tiefer selbst als der der Geometrie? " (Frege, 1884:44).

notion of 'distance' that involves number, and since a line is a spatial figure extended in 1-dimension, it clearly only has a determinate meaning in subjection to the first order of spatial extension. We have argued that in both domains (number and space) there is a strict correlation between the law side and the factual side. In the case of space it is therefore possible to discern a reference to number both at the law side and the factual side. Speaking of one or more dimensions presupposes the meaning of number on the law side and specifying the one dimensional extension (magnitude) of something like a line presupposes the meaning of the number employed in the designation of the *length* of the line. The domain of number therefore appears to be more basic because an analysis of the meaning of space invariably calls upon foundational arithmetical considerations.

This conclusion is further supported by the approach of Maddy where she argues that most recent textbooks "view of set theory as a foundation of mathematics" (Maddy, 1997:22) and that a set theoretic foundation can "isolate the mathematically relevant features of a mathematical object" in order to find a "set theoretic surrogate" for those features (Maddy, 1997:27, 34).[15] Bernays categorically asserts that "the representation of number is more elementary than geometrical representations" (Bernays, 1976:69, see also page 75: "For our human understanding the concept of number is more immediate than the representation of space"). In general one may view the arithmeticism of Weierstrass, Dedekind and Cantor as an (over-estimated) acknowledgement of the foundational position of the domain of number.

3.1.2 Interconnections between functional domains

A metaphorical way to capture this state of affairs is to use an image from human memory by saying that within the meaning of space (both at the law side and the factual side), we discover configurations *reminding* us of the core meaning of number. A key element in all metaphorical descriptions is found in the connection between *similarities* and *differences*. Whenever what is different is shown in what is similar, one may speak about *analogies*. But we want to broaden the scope of an analogy in order to include more than what is normally accounted for in a theory of

15 Already in 1910 Grelling recognized set theory as the foundation of mathematics as a whole: "Zuerst ausgebildet als Hilfsmittel der Untersuchung bei gewissen Fragen der Analysis, hat sich die under den Händen inhres Schöpfers Georg Cantor und sein Schüler zu einer selbständigen metahmatischen Disziplin entwickelt, die heute die Grundlage der gesamten Mathematik bildet" ("In the first place developed as an auxilliary tool for the investigation of certain questions of analysis [set theory] in the hands of Cantor and his pupils [it was] developed into an independent mathematical discipline. Currently it constitutes the foundation of mathematics in its entirety" (Grelling, 1910:6).

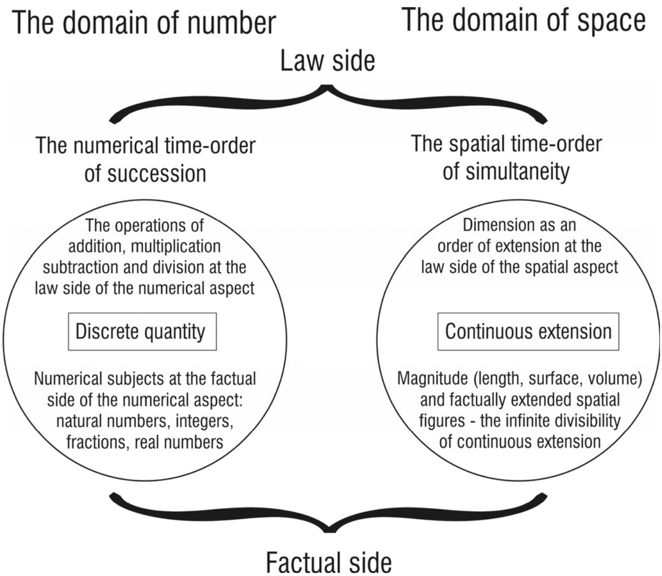
metaphor. Our first designation already achieves this goal, for whenever *differences* between entities and properties bring to expression what is *similar* between those entities or properties, we meet instances of an *analogy*.[16] Implicit in the nature of an analogy is the distinction between something *original* and something else which 'reminds' one of what is originally given but which is now encountered in a *non-original* context, i.e., within an *analogical* setting. This is exactly what we have noticed in the terms 'distance' and 'dimension' – for in both cases we are *reminded* of the quantitative meaning of number. In terms of the idea of an analogy one can say that there is an analogy of number on the law side of the spatial aspect (one, two, three or more dimensions) and that there is an analogy of number at the factual side of the spatial aspect (magnitude – as the correlate of different orders of extension: in *one* dimension magnitude appears as length, in *two* dimensions it appears as area, in *three* as volume, etc.). An account of the basic position of number can now be articulated in terms of the idea of analogies, for since basic *numerical analogies* are presupposed within the domain of space, the original meaning of number is indeed foundational for the meaning of space.

The attentive reader would have noticed that in the previous paragraph we have introduced a new word in order to refer to the domains of number and space, namely the term 'aspect'. The underlying hypothesis of this usage is found in the theory that the various aspects of reality belong to a distinct dimension which is fundamentally different from the concrete *what-ness* of (natural and social) entities (such as things, plants, animals, artifacts, societal collectivities and human beings). These concrete entities (and the processes in which they are involved) all function within the different aspects of reality. Questions about the way in which entities exist concern their *how-ness*, their mode of being. Aspects in this sense are therefore *modes of being*. That my chair is *one* and has *four* legs reveal its function within the quantitative mode of reality; that it has a certain *shape* and *size* highlights its spatial function; that it has a certain economic value demonstrates its function within the economic mode of reality, and so on. This dimension of functions or aspects can also be designated as that of *modalities* or *modal functions*. What has already been said about the domains of number and space concern properties that may serve to define

16 Whenever *entities* are involved in the figurative speech we employ such designations are considered to be *metaphorical*. But as soon as similarities and differences between modal functions (as they will be explained below) are captured, these purely aspectual interrelations represent a domain of analogies distinct from metaphors. When purely intermodal connections (analogies) are metaphorically explored, an element of the entitary dimension of reality will always be present (such as it is found in the metaphor of a person being 'reminded' of an original domain).

the nature of an aspect:[17] every aspect contains a sphere of modal (functional) laws (at its law side); a factual side (subjected to modal laws); and a core meaning qualifying, characterizing or stamping all the structural moments discernable within an aspect (in particular also the analogical elements pointing to the meaning of other modal functions of reality). This core meaning or meaning-nucleus guarantees the uniqueness and irreducibility of every aspect and it underlies the inevitable use of primitive (= indefinable) terms by those disciplines that explore a specific modal aspect as angle of approach to reality.

Some of these structural features are captured in the following sketch:



4. Philosophical implications

Within the quantitative aspect, for example, the order of succession on its law side lies at the basis of arithmetical operations (such as addition and multiplication and their inverses) and it makes possible our basic numeri-

17 Note that any description of modal aspects inevitably employs *metaphors* (involving entitary analogies). Fore example, one may say that aspects are 'points of entry' to reality, that they provide an 'angle of approach' to reality, and so on. Conversely, the modal aspects provide access to the dimension of entities – they may serve as *modes of explanation* of concrete reality.

cal awareness of *greater* and *lesser*.[18] Within space this awareness of greater and lesser may occur in the context of dimensional extension (for example as the vertical opposition of *higher* and *lower*). In their metaphor theory Lakoff and Johnson do not consider ontic aspects and their interconnections exemplified in modal analogies, for they restrict themselves to mappings between "conceptual domains." Consequently, they look at quantity and verticality in terms of "the associations between More and Up and [48] between Less and Down" which "constitute a cross-domain mapping between the sensorimotor concept of verticality (the source domain) and the subjective judgment of quantity" (the target domain) (Lakoff & Johnson, 1999:47-48). However, we have argued that the nature of *dimensional extension* is not *purely* spatial because inherently it reveals numerical analogies (both at the law side and the factual side of this aspect). Therefore the notion of *verticality* is embedded in that of *dimensionality* – and the latter collapses into nothingness outside its coherence with the (foundational) quantitative meaning of *one*, *two* and *three*. Whereas the "conceptual domains" of Lakoff and Johnson may be *disconnected*, the ontic nature of the aspects of number and space displays an unbreakable connectedness which is seen in the modal analogies that underscore the fact that the unique meaning of an aspect comes to expression through its coherence with other aspects.[19]

4.1 The various academic disciplines are dependent on philosophical presuppositions

Whereas the various academic disciplines (special sciences) in general approach reality through the gateway of specific modal aspects, it belongs

18 The arithmetical order of succession determines our most basic intuition of infinity, in the literal sense of one, another one, and so on, without an end, *endlessly*, *indefinitely*, *infinitely*. The traditional designation of this kind of infinity, known as the *potential infinite*, lacks an intuitive appeal. But when we alternatively refer to the 'successive infinite' this shortcoming is left behind. The other kind of infinity, traditionally known as die *actual infinite*, also calls for an 'intuitively transparent' designation (see note 26 below). The successive infinite, presupposed in the infinite divisibility of continuity, makes possible induction, which, according to Weyl, guarantees that mathematics does not collapse into an enormous tautology (Weyl, 1966:86). According to Gödel non-"tautological" relations between mathematical concepts "appears above all in the circumstance that for the primitive terms of mathematics, axioms must be assumed" (1995:320-321). In the case of finitism where the "general concept of a set is *not* admitted in mathematics proper … induction must be assumed as an axiom" (Gödel, 1995:321).

19 In a more recent work their understanding of "conceptual metaphor" brought Lakoff and Núñez to the view that *continuity* and *discreteness* actually are *opposites* (Lakoff & Núñez, 2000:324) – in stead of merely being uniquely different but mutually cohering modal aspects of reality. Opposites occur *within* aspects (like 'high' and 'low' within the spatial aspect), but not *between* aspects.

to the task of philosophy to account for the uniqueness and coherence of these different modes of explanation as a special case of the basic (philosophical) problem of unity and diversity.[20] The questions asked in the title of this article paved the way for this insight, because both in connection with the claim that a line is the ‘distance’ between two points and in connection with the difference between arithmetical addition (2+2=4) and spatial addition ($\underline{2}+\underline{2}=\sqrt{8}$) our considerations invariably were confronted with two different aspects, namely those of number and space.[21] But the question regarding the mutual relationship between number and space caused a dialectical movement to and fro between the extremes of arithmeticcism and geometricism. The philosophical importance of considering more than one aspect therefore first of all stems from the history of mathematics, for we have noted that the initial Pythagorean tendency to arithmetize was followed by a geometrization which, since Cauchy and Weierstrass, once again reverted to an assumed arithmetization (in modern set theory).

At this point Fern’s mentioned claim that “[M]athematical calculations are paradigmatic instances of universally accessible, rationally compelling argument” (Fern, 2002:96-97) may be contrasted with the assessment of

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20 In passing we may mention the different modes of explanation employed during the history of our understanding of matter. In the early modern era the Greek focus on *number* (the Pythagoreans) and the switch to *space* was followed by the exploration of the *kinematic* mode of explanation. As primary qualities of matter Galileo considers arithmetical properties (countability), geometrical properties (form, size, position, contact) and kinematical properties (movement). Hucklenbroich writes: “G. Galilei zählt als primäre Qualitäten der Materie arithmetische (Zählbarkeit), geometrische (Gestalt, Größe, Lage, Berührung) und kinematische Eigenschaften (Beweglichkeit) auf” (Hucklenbroich, 1980:921). Only at the beginning of the 20th century modern physics came to peace with the term *force*. Hertz still believed that he had to reject the (physical) concept force, claiming that it is something inherently antinomic (cf. Katscher, 1970:329) – a view similar to the one found in Bertrand Russell’s work: *Principles of Mathematics*. The only difference is that Russell speaks about ‘force’ as a “mathematical fiction”: “The first thing to be remembered is – what physicists now-a-days will scarcely deny – that force is a mathematical fiction, not a physical entity” (Russell, 1956:482; cf. 494 ff.). The different ‘Meßgrößen’ (units of measurement) distinguished by Lorenzen in his ‘proto-physics’, are also strictly correlated with the spatial, kinematical and physical contexts in which such magnitudes appear: length, duration, mass and charge (see Lorenzen, 1976:1 ff.).

21 Gödel provided his own account of the dimension of modal aspects. Next to a physical causal context within which something can be 'given', he refers to data of a second kind which are open to 'semiperceptions'. He says that they are not something “purely subjective as Kant says.” Rather they, too, “may represent ‘an aspect of objective reality’, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality (quoted by Wang, 1988:304). Bernays also distinguishes between more than one kind of factuality - entitary (called ‘concrete’ by him) and modal-functional (he preferably speaks about ‘idealization’ in the latter case) (Bernays, 1976:122).

Kline two decades earlier:

> The developments in the foundations of mathematics since 1900 are bewildering, and the present state of mathematics is anomalous and deplorable. The light of truth no longer illuminates the road to follow. In place of the unique, universally admired and universally accepted body of mathematics whose proofs, though sometimes requiring emendation, were regarded as the acme of sound reasoning, we now have conflicting approaches to mathematics. Beyond the logicist, intuitionist, and formalist bases, the approach through set theory alone gives many options. Some divergent and even conflicting positions are possible even within the other schools. Thus the constructivist movement within the intuitionist philosophy has many splinter groups. Within formalism there are choices to be made about what principles of metamathematics may be employed. Non-standard analysis, though not a doctrine of any one school, permits an alternative approach to analysis which may also lead to conflicting views. At the very least what was considered to be illogical and to be banished is now accepted by some schools as logically sound (Kline, 1980:275-276).[22]

Like every other academic discipline mathematics will always be confronted with the philosophical problems of uniqueness and coherence (unity and diversity). One facet of this foundational philosophical problem is given in the unavoidability of employing analogical terms, i.e., in the use of terms reflecting the interconnection between different aspects. This follows from the fact that different modal aspects are interrelated in such a way that everyone,

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What Kline states does not concern diverging views of mathematics but mathematics itself. Compare two further statements. Kleene remarks: “The intuitionists have created a whole new mathematics, including a theory of the continuum and a set theory. This mathematics employs concepts and makes distinctions not found in the classical mathematics” (Kleene, 1952:52). And Beth says: “It is clear that intuitionistic mathematics is not merely that part of classical mathematics which would remain if one removed certain methods not acceptable to the intuitionists. On the contrary, intuitionistic mathematics replaces those methods by other ones that lead to results which find no counterpart in classical mathematics” (Beth, 1965:89).

22. This distinction between physical space and mathematical space - and about the infinite divisibility of what is continuous is not concerned with the mathematical (set theoretical) understanding of continuity, for in the latter case two criteria are applied. Cantor, for example defines 'continuity' in terms of a perfectly coherent set (see Strauss, 2002:12-18) - where the feature of 'coherence' is equivalent to the denseness of a set in itself (see Natanson, 1960:37). The Cantorian definition of a continuous set as being perfectly coherent (Cantor, 1962:194) is also explained by Klein (see Klein, 1928:105).
“Erst durch die zeitherige Entwicklung der Geometrie und der Physik tritt die Notwendigkeit hervor, zwischen dem Raum als etwas Physikalischem und dem Raum als eine ideellen, durch geometrische Gesetze bestimmten Mannigfaltigkeit zu unterscheiden” (Bernays, 1976:37). (“Only through the contemporary development of geometry and physics did it become necessary to distinguish between space as something physical and space as an ideal multiplicity determined by spatial laws.”)

within its own structure, reflects the modal meaning of others. Physical extension, for example, shows some likeness with spatial extension. However, in this moment of similarity, the modal difference is simultaneously expressed – spatial extension is continuous in the sense that it allows for an infinite divisibility, whereas physical space is not continuous (since it is determined by the quantum-structure of energy) and is therefore not infinitely divisible (already in 1925 Hilbert mentioned this difference – see Hilbert, 1925:164).[23] Bernays also distinguishes between physical space and mathematical space.[24] Sensitive space, for example the sensitivity for distinct sensations on the human skin, may be experienced as continuous in spite of the fact that the stimuli are *physically* discontinuous (distinct) (see Gosztonyi, 1976, I:13).

4.2 The primitive meaning of space underlying Hilbert's primitive terms

Within the arithmetical aspect the factual relation between numbers is constituted as subject-subject relations – as it is present in the addition of numbers, the multiplication of numbers or establishing the numerical difference between numbers (subtraction). However, at the factual side of the spatial aspect there are not only subject-subject relations (such as intersecting lines), for there are also subject-object relations present, mainly expressed in the idea of a *boundary*.

Already in his abstraction theory Aristotle employed the notion of a boundary (or limit) – which is intuitively immediately associated with *spatial* notions (Aristotle used the term *eschaton*). By the 13th century AD Thomas Aquinas accounts for a 1-dimensional line by means of a descending series of abstractions. In contradistinction to natural bodies, all mathematical figures are infinitely divisible. The Aristotelian legacy is clearly seen in his definition of a point as the *principium* of a line (cf. *Summa*

23. See also Brouwer, 1924:554. When a "species" π does not contain a continuum as part it is of dimension 0 in the Menger-Urysohn sense.
It is therefore unjustifiable to see a line as a set of points. But it falls outside the scope of this article to highlight the circularity present in Grünbaum's attempt to argue for a consistent conception of the extended linear continuum as an aggregate of unextended elements (see Grünbaum, 1952). Grünbaum did not realize that the actual infinite – or, as one may prefer to call it: the at once infinite – depends upon a crucial spatial feature, namely the order of simultaneity. In the idea of the at once infinite the meaning of number analogically points towards the meaning of space.

24. The system of rational numbers therefore represents an anticipatory analogy at the factual side of the numerical aspect to the factual whole-parts relation within the spatial aspect.
The combined perspective thus obtained actually underlies the remark that within the system of rational numbers we encounter an anticipation to a retrocipation. The divisibility of an interval points forward to (anticipates) the factual spatial whole-parts relation, whereas the latter (with its infinite divisibility) points backward (retrocipates) to the order of succession on the law side of the numerical aspect.

Theologica, I,II,2), which indicates the fact that a determinate line-stretch has points at its extremities ("cuius extremitates sunt duo puncta" – *Summa Theologica*, I,85,8). This legacy returns in a somewhat more general form in the 18th century (the era of the *Enlightenment*). Kant remarks:

> Area is the boundary of material space, although it is itself a space, a line is a space which is the boundary of an area, a point is the boundary of a line, although still a position in space (Kant, 1783, A:170).

In 1912 Poincaré discussed similar problems. Concerning the way in which geometers introduce the notion of three dimensions he says: "Usually they begin by defining surfaces as the boundaries of solids or pieces of space, lines as the boundaries of surfaces, points as the boundaries of lines" (cf. Hurewicz & Wallman, 1959:3). Although only related to three dimensions, Poincaré here provides us with an intuitive approach to dimension, implicitly stressing the unbreakable correlation between the law side and the factual side in the spatial aspect:

> ... if to divide a continuum it suffices to consider as cuts a certain number of elements all distinguishable from one another, we say that this continuum is of one dimension; if, on the contrary, to divide a continuum it is necessary to consider as cuts a system of elements themselves forming one or several continua, we shall say that this continuum is of several dimensions (Hurewicz & Wallman, 1959:3).

Before 1911 the problem of dimension was confronted with two astonishing discoveries. Cantor showed that the points of a line can be correlated one-to-one with the points of a plane, and Peano mapped an interval continuously on the whole of a square. The crucial question was whether, for example, the points of a plane could be mapped onto the points of an interval in both a continuous and one-to-one way. Such a mapping is called homeomorphic. The impossibility to establish a homeomorphic mapping between a "m-dimensional set and a (m+1)-dimensional set ($h > 0$)" was solved by Lüroth for the case where $m \leq 3$ (Brouwer, 1911:161; cf. also the footnote on page 161). Brouwer provided the first general proof of the invariance of the number of a dimension (see Brouwer, 1911:161-165). Exploring suggestions of Poincaré, Brouwer in 1913 introduced a precise (*topologically invariant*) definition of *dimension*, which was independently recreated and improved in 1922 by Menger and Urysohn (cf. Hurewicz & Wallman, 1959:4). Menger's formulation (still adopted by Hurewicz and Wallman) simply reads:

a) the empty set has dimension -1,
b) the dimension of a space is the least integer n for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than n (Hurewicz & Wallman, 1959:4, cf. p.24).[25]

25 See also Brouwer, 1924:554. When a "species" π does not contain a continuum as part it is of dimension 0 in the Menger-Urysohn sense.

Whereas a spatial subject is always factually extended in some dimension (such as a 1-dimensional line, a 2-dimensional area, and so on), a spatial object merely serves as a *boundary* (in a delimiting way). The boundaries of a determined line-stretch are the two points delimiting it (with the line as a one-dimensional spatial subject). But these boundary points themselves are not extended in one dimension. Within one dimension points are therefore not spatial subjects but merely spatial objects, dependent upon the factual extension of the line. Yet a line may serve in a similar delimiting way within two dimensions – for the lines delimiting an area are not themselves extended in a two dimensional sense. In a similar fashion a surface can assume the role of a spatial object, namely when it delimits three dimensional spatial figures (such as a cube).

In general it can therefore be stated that whatever is a spatial subject in *n* dimensions is a spatial object in *n+1* dimensions. A point is a spatial object in one dimension (an objective numerical analogy on the factual side of the spatial aspect), and therefore a spatial subject in *no* dimension (i.e., in *zero* dimensions). In terms of the fundamental difference between a spatial subject and a spatial object, it is impossible to deduce spatial extension from spatial objects (points).[26]

We can now account for the three primitive terms in Hilbert's axiomatization of geometry in terms of the spatial subject-object relation. The term 'line' reflects the primary existence of a (one dimensional) spatial subject, the term 'point' highlights the primary existence of a (one dimensional) spatial object and the phrase 'lies on' accounts for the *relation* between a spatial subject and a spatial object – in other words, for the *spatial subject-object relation*.

Primitive features at the factual side of the spatial aspect	Subject	Object	Relation
The primitive terms in Hilbert's axiomatization of geometry (1899)	Line	Point	Lies on

26 It is therefore unjustifiable to see a line as a set of points. But it falls outside the scope of this article to highlight the circularity present in Grünbaum's attempt to argue for a consistent conception of the extended linear continuum as an aggregate of unextended elements (see Grünbaum, 1952). Grünbaum did not realize that the actual infinite – or, as one may prefer to call it: the at once infinite – depends upon a crucial spatial feature, namely the order of simultaneity. In the idea of the at once infinite the meaning of number analogically points towards the meaning of space.

5. Concluding remark

Our analysis of the meaning of space in its coherence with number and of the difference between numerical and spatial addition highlighted the insight that the domains of number and space are distinct and that the meaning of space depends upon the meaning of number. The interconnections involved between these two aspects entail fundamental philosophical problems which even caused diverging orientations within mathematics as a discipline. Of course there are more interconnections between number and space than those highlighted in this article. Without entering in a detailed analysis a mere hint will be given of what these other interconnections entail.

Given the foundational position of the numerical aspect in respect of the spatial aspect one should also differentiate between analogies pointing backward and forward between these two aspects. Distance, for example, at the factual side of the spatial aspect points backwards to the numerical mode. It may therefore be designated as a *retrocipatory* analogy within space. The idea of the at once infinite (see note 25), furthermore, represents an *anticipatory* analogy on the law side of the numerical aspect pointing towards the order of simultaneity on the law side of the spatial aspect. Likewise, the infinite divisibility of any (factually extended) spatial subject refers back to the law side of the numerical aspect, where the order of arithmetical succession reveals the primitive meaning of endlessness (see note 17). In this context one should also mention that the continuous extension of any spatial subject embodies the original meaning of the spatial *whole-parts relation* (with its implied infinite divisibility). The *interval* within the system of rational numbers analogically reflects this infinite divisibility of a spatial subject,[27] and the latter, as we have just pointed out, represents a retrocipation from space to the primitive meaning of the successive infinite on the law side of the numerical aspect.[28] Only when the anticipatory analogies within number are opened up and explicitly explored (as it was done by Weierstrass, Cantor and Dedekind) is it possible to arrive at a meaningful (spatially disclosed or anticipatory) treatment of the real number system. Only then is it possible to explain why the sub-discipline known as real analysis should therefore actually be

27 The system of rational numbers therefore represents an anticipatory analogy at the factual side of the numerical aspect to the factual whole-parts relation within the spatial aspect.

28 The combined perspective thus obtained actually underlies the remark that within the system of rational numbers we encounter an anticipation to a retrocipation. The divisibility of an interval points forward to (anticipates) the factual spatial whole-parts relation, whereas the latter (with its infinite divisibility) points backward (retrocipates) to the order of succession on the law side of the numerical aspect.

seen as a *spatially disclosed numerical theory* – and not merely as a *purely* arithmetical theory. But a thorough analysis of all these additional perspectives cannot be dealt with in this context.

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